Reliability Computation of a Dynamic Stress-Strength Model with Random Cycle Times

K. C. Siju\textsuperscript{1} and M. Kumar\textsuperscript{2}

\textsuperscript{1,2}Department of Mathematics, National Institute of Technology, Calicut, India
\textsuperscript{2}mahesh@nitc.ac.in
\textsuperscript{1}sijulal732@gmail.com

Abstract

Stress-Strength models of determining the reliability of a product depend on its physical (internal) strength surviving (being greater than) the applied loads. The inclusion of time factor by considering the application of loads over time initiates the development of time-dependent stress-strength reliability. In this paper, we study the dynamic stress-strength reliability for (i). Random-fixed stress, and strength (ii). Random-fixed strength and random independent stress. The expressions for the reliability of a component are obtained for initial exponential and Weibull, stress and strength random variable. The stresses are repeatedly applied at random cycle times and the number of cycles in an interval is assumed to follow a Poisson distribution. Finally, graphical illustrations are presented using MATLAB programming for the n component parallel system.

AMS Subject Classification: 62P30, 62N05, 90B25.
Key Words and Phrases: Time Dependent; Stress; Strength; Random Cycle Times; Reliability.

1 Introduction

Immense research of stress-strength reliability models concentrates on the evaluation of the reliability of a product based on the application of single load (stress) (see, [4]). In reality, the product may be acted upon by several stresses at different time points (see,[6]). This requires the knowledge of the frequency of occurrence of stresses and the time of application of stress. These behaviours, characterise the repeated stresses. Deterministic and random cycle times are the basic classifications
of the time of stress occurrence. The nature of stress and strength variables may also be classified into deterministic, random-fixed and random-independent (see, [5]). There arise several situations of reliability computations with regard to the classification of time of application of stresses and nature of random variables. Some of the studies related to such models were performed by Gopalan and Venkateswarlu, and Eryilmaz (see, [2], [3], [1]). One of the situation is discussed by Siju and Kumar (see, [7]). Here we discuss two more cases of these combinations. The rest of the paper is organised as follows. In Section 2, the two cases are discussed for initial exponential and Weibull stress and strength. Section 3 provides the illustration of our findings for a parallel configuration and section 4 gives the conclusions.

2 Reliability Computation

2.1 Random-fixed Strength and Random-fixed Stress

Suppose a component is acted upon by stress which is applied repeatedly at random cycle times. The times at which the stresses are applied repeatedly are called cycle times. Let the strength of a component in the $j^{th}$ cycle be $X_j = X_{j-1} - a, j = 1, 2, ... i$ where $a \geq 0$ is a known constant. Let the stress acting on a component in the $j^{th}$ cycle be $Y_j = Y_{j-1} + b, j = 1, 2, ... i$ where $b \geq 0$ is also a known constant. Let the probability density function of the initial strength $X_0$ be $f(x_0)$. Let the probability density function of the initial stress $Y_0$ be $g(y_0)$. The reliability of a component under random cycle times is given by

$$R(t) = \sum_{i=0}^{\infty} p_i(t) R_i,$$

where, $p_i(t)$ is the probability that $i$ cycles occur in the time interval $[0,t]$ and $R_i$ is the probability of no failure after $i$ applications of the stress ([5]). Let $E_i$ be the event that no failure ($X_i > Y_i$) occurs on the $i^{th}$ cycle. Now the probability after $i$ cycles is given by

$$R_i = P(E_1 \cap E_2 \cap E_3 \cdots \cap E_i)$$

$$= P((X_1 > Y_1) \cap (X_2 > Y_2) \cap (X_3 > Y_3) \cdots \cap (X_i > Y_i))$$

$$= P(((x_0 - a) > (y_0 + b)) \cap ((x_0 - 2a) > (y_0 + 2b)) \cap ...$$

$$\cap ((x_0 - ia) > (y_0 + ib)))$$

$$= P(x_0 > (y_0 + ia + b)))$$

$$= \int_{-\infty}^{\infty} \int_{y_0 + i(a+b)}^{\infty} f(x_0)g(y_0)dxdy. \hspace{1cm} (2)$$

The time dependent stress-strength reliability is

$$R(t) = \sum_{i=0}^{\infty} \int_{-\infty}^{\infty} \int_{y_0 + i(a+b)}^{\infty} f(x_0)g(y_0)dxdy. \hspace{1cm} (3)$$
2.1.1 Initial Exponential stress and strength

Let the initial strength $X_0$ and the initial strength $Y_0$, follow exponential distribution given by $f(x_0) = \lambda e^{-\lambda x_0}, \lambda > 0, x_0 > 0$ and $g(y_0) = \mu e^{-\mu y_0}, \mu > 0, y_0 > 0$. Let the number of cycles occurring in a given time interval follow Poisson distribution given by

$$p_i(t) = \frac{e^{-\alpha t}(\alpha t)^i}{i!}, i = 0, 1, 2, \ldots, (4)$$

where, $\alpha$ is a parameter representing the mean number of occurrences of $i$ cycles per unit time. Using equation (2), the probability of success of all $i$ cycles is $R_i = e^{-i\lambda(a+b)}$, for $i = 1, 2, \ldots$. Note that $R_0 = P(X_0 > Y_0) = \frac{\mu}{\lambda + \mu}$. The time dependent stress-strength reliability of a single component is obtained to be

$$R(t) = e^{-\alpha t}[e^{(1-e^{-\lambda(a+b)})} - \frac{\lambda}{\lambda + \mu}] (5)$$

Using equation (5) the expression for system reliability $R_s(t)$, for the following configurations are given as follows,

- **n - component parallel system - Identical components:**

$$R_s(t) = 1 - (1 - R(t))^n$$

$$= 1 - (1 - e^{-\alpha t}[e^{(1-e^{-\lambda(a+b)})} - \frac{\lambda}{\lambda + \mu}])^n$$

(6)

- **n - component parallel system - Non-Identical components:**

$$R_s(t) = 1 - \prod_{j=1}^{n}(1 - R_j(t))$$

$$= 1 - \prod_{j=1}^{n}(1 - e^{-\alpha t}[e^{(1-e^{-\lambda_j(a+b)})} - \frac{\lambda_j}{\lambda_j + \mu_j}])$$

(7)

- **n - component series system - Identical components:**

$$R_s(t) = (R(t))^n = (e^{-\alpha t}[e^{(1-e^{-\lambda(a+b)})} - \frac{\lambda}{\lambda + \mu}])^n$$

(8)

- **n - component series system - Non-Identical components:**

$$R_s(t) = \prod_{j=1}^{n}(R_j(t)) = \prod_{j=1}^{n}(e^{-\alpha t}[e^{(1-e^{-\lambda_j(a+b)})} - \frac{\lambda_j}{\lambda_j + \mu_j}])$$

(9)

International Journal of Pure and Applied Mathematics Special Issue
2.1.2 Initial Weibull stress and strength

Let the initial strength $X_0$ and initial stress $Y_0$ follow Weibull distribution given by $f(x_0) = \frac{\gamma \beta}{\alpha} \left(\frac{x_0}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x_0}{\alpha}\right)^\beta}$, $\gamma > 0$, $\beta > 0$, $x_0 > 0$ and $g(y_0) = \theta \beta (y_0)^{\beta-1} e^{-\theta (y_0)^\beta}$, $\theta > 0$, $\beta > 0$, $y_0 > 0$. Using equations (4) and (1), the probability of success of all the $i$ cycles is given by $R_i = e^{-\gamma(i(a+b))^\beta}$, for $i = 1, 2, \ldots$, and $R_0 = P(X_0 > Y_0) = \frac{\theta}{\gamma + \theta}$.

The time dependent stress-strength reliability of a single component is obtained to be

$$R(t) = e^{-\alpha t} \frac{\theta}{\gamma + \theta} + \sum_{i=1}^{\infty} \frac{e^{-\alpha t} (\alpha t)^i}{i!} e^{-\gamma(i(a+b))^\beta}$$

$$= e^{-\alpha t} \left[ \frac{\theta}{\gamma + \theta} + \sum_{i=1}^{\infty} \frac{(\alpha t)^i}{i!} e^{-\gamma(i(a+b))^\beta} \right] \quad (10)$$

The closed form expressions for system reliability $R_s(t)$, for the configurations mentioned in section 2.1.1 are obtained in a similar manner.

2.2 Random-fixed Strength and Random independent Stress

Let the strength of a component in the $j$th cycle be $X_j = X_{j-1} - a, j = 1, 2, \ldots i$ where $a \geq 0$ is a known constant. Let the stress acting on the component in the $j$th cycle be $Y_j, j = 1, 2, \ldots i$. Assume that $Y_1, Y_2, \ldots, Y_i$ are independent with probability density function $g(y_j), j = 1, 2, \ldots i$. Let the probability density function of the initial strength $X_0$ be $f(x_0)$. Now the probability after $i$ cycles is given by

$$R_i = P(E_1 \cap E_2 \cap E_3 \cdots \cap E_i)$$

$$= P((X_1 > Y_1) \cap (X_2 > Y_2) \cap (X_3 > Y_3) \cdots \cap (X_i > Y_i))$$

$$= P((x_0 > (y_1 + a)) \cap (x_0 > (y_2 + 2a)) \cdots \cap (x_0 > (y_i + ib)))$$

$$= P((y_1 < (x_0 - a)) \cap (y_2 < (x_0 - 2a)) \cdots \cap (y_i < (x_0 - ia)))$$

$$= \prod_{j=1}^{i} P(y_j < (x_0 - ja)) = \prod_{j=1}^{i} \int_{-\infty}^{\infty} \int_{y_j + ja}^{\infty} f(x_0) g(y_j) dx_0 dy_j.$$

The time dependent stress-strength reliability is

$$R(t) = \sum_{i=0}^{\infty} p_i(t) \prod_{j=1}^{i} \int_{-\infty}^{\infty} \int_{y_j + ja}^{\infty} f(x_0) g(y_j) dx_0 dy_j. \quad (11)$$

2.2.1 Initial exponential strength and independent exponential stress

Let the initial strength random variable $X_0$ follow exponential distribution given by $f(x_0) = \lambda e^{-\lambda x_0}, \lambda > 0, x_0 > 0$. Let the stress random variable $Y_j, j = 1, 2, \ldots i$ follow exponential distribution given by $g(y_j) = \mu e^{-\mu y_j}, \mu > 0, y_j > 0$. The initial probability of success $R_0 = \frac{\mu}{\lambda + \mu}$. Now the probability of success in the first cycle
is given by $R_1 = \int_{-\infty}^{\infty} \int_{y_1+a}^{\infty} f(x_0) g(y_1) dx_0 dy_1 = e^{-\lambda(a)}$. Similarly $R_2$ is obtained to be $e^{-\lambda(2a)}$ and so on. Therefore the probability of success after $i$ cycles of is given by $R_i = \prod_{j=1}^{i} e^{-\lambda(ja)}$. Assuming Poisson number of cycle occurrences in a given time interval, the time dependent stress-strength reliability of a single component is given by

$$R(t) = e^{-\alpha t} \left[ \frac{\mu}{\lambda + \mu} + \sum_{i=1}^{\infty} \frac{(\alpha t)^i}{i!} \prod_{j=1}^{i} e^{-\lambda(ja)} \right]$$  \hspace{1cm} (12)$$

Equation (12) may be used further to obtain the system reliability of various configurations as in section 2.1.1.

### 2.2.2 Initial Weibull strength and independent Weibull stress

Let the initial strength $X_0$ follow Weibull distribution given by $f(x_0) = \gamma \beta (x_0)^{\beta-1} e^{-\gamma x_0^\beta}$, $\gamma > 0, \beta > 0, x_0 > 0$. Let the stress $Y_j, j = 1, 2, ..., i$ follow Weibull distribution given by $g(y_j) = \theta \beta (y_j)^{\beta-1} e^{-\theta y_j^\beta}$, $\theta > 0, \beta > 0, y_j > 0$. The probability of success of all the $i$ cycles is given by $R_i = \prod_{j=1}^{i} e^{-\gamma(ja)^\beta}$, for $i = 1, 2, ..., \text{and} R_0 = \frac{\alpha}{\gamma + \theta}$. The time dependent stress-strength reliability of a single component is given by

$$R(t) = e^{-\alpha t} \left[ \frac{\theta}{\gamma + \theta} + \sum_{i=1}^{\infty} \frac{(\alpha t)^i}{i!} \prod_{j=1}^{i} e^{-\gamma(ja)^\beta} \right]$$ \hspace{1cm} (13)$$

The expressions for system reliability of the various configurations may be obtained as in section 2.1.1.

### 3 Illustration - Exponential model - Parallel system

As an illustration, we evaluate the time-dependent stress-strength reliability for a parallel system with four identical components. The calculations are performed for the random-fixed stress, and strength model. Evaluations for the second case may be performed similarly. Let the values of the constants, and strength-stress parameters be $a = 0.1, b = 0.3, \lambda = 0.3$ and $\mu = 0.02$. Using MATLAB programming the expressions of system reliability $(R_{sk}(t), k = 1, 2, 3)$, for the various values of mean number of Poisson cycles $\alpha = 2, 10, 20$ are evaluated to be, $R_{s1}(t) = 1.0 - 1.0 \ast (exp(-20.0 \ast t) \ast (exp(1.774 \ast t) + 0.0625) - 1.0)^4$, $R_{s2}(t) = 1.0 - 1.0 \ast (exp(-10.0 \ast t) \ast (exp(8.869 \ast t) + 0.0625) - 1.0)^4$, and $R_{s3}(t) = 1.0 - 1.0 \ast (exp(-20.0 \ast t) \ast (exp(17.74 \ast t) + 0.0625) - 1.0)^4$. The graphical representation is given in Figure 1-(b).

### 3.1 Weibull model

The illustration for the parallel system with four identical components for the Weibull initial strength and stress model is performed in a similar manner as done for the ‘Exponential model’. Also the Weibull scale parameter $\gamma$ takes the same values of $\lambda$ given in the ‘Exponential model’ and the values of shape parameter
The graphical representation of system reliability for the specific parallel system with identical components is given in Figure 1. The calculations for other configurations are performed in a similar manner.

4 Conclusions

This paper studies the time-dependent stress-strength reliability for two combinations of the nature of stress and strength random variables. For the random-fixed nature of the random variable, we have assumed the stress and strength of a unit in the present cycle to be dependent on the same in the previous cycle. We have presented the system reliability expression (see, equations 5, 10, 12 and 13) for a single component in the presence of Poisson number of random cycle occurrences and, initial exponential, Weibull strength and stress. The system reliability expressions for various configurations of the first case are presented in equations 6, 7, 8 and 9. The graphical representation of the Weibull initial strength and stress in Figure 1 shows that for increasing number of cycles of stresses, the system reliability decreases slowly, gradually and rapidly for \( \beta = 0.7, 1, 2 \) respectively. The graphical representation for exponential strength and stress is given in Figure 1-(b). Better performance of system reliability is visualised in the graphical representation when \( \beta = 0.7 \). This implies that a broader distribution of initial strength and stress gives increasing values of system reliability.

Figure 1: Random fixed stress, and strength : Reliability of a parallel system having four identical components with Weibull initial strength, and stress (c) - \( \beta = 0.7, 2 \) - and (d) - \( \beta = 1 \) represents the case of exponential initial strength, and stress.

References


