Lourdusamy’s Conjecture on Shadow Graphs of Path

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Abstract

Given a distribution of pebbles on the vertices of a connected graph \(G\), a pebbling move is defined as the removal of two pebbles from some vertex and the placement of one of those pebbles on an adjacent vertex. The \(t\)-pebbling number \(f_t(G)\) of a simple connected graph \(G\) is the smallest positive integer such that for every distribution of \(f_t(G)\) pebbles on the vertices of \(G\), we can move \(t\) pebbles to any target vertex by a sequence of pebbling moves. Graham conjectured that for any connected graphs \(G\) and \(H\), \(f(G \times H) \leq f(G)f(H)\). Lourdusamy further conjectured that \(f_t(G \times H) \leq f(G)f_t(H)\), for any positive integer \(t\). In this paper we show that Lourdusamy’s conjecture is true, when \(G\) is a shadow graph of path and \(H\) is a graph satisfying the \(2t\)-pebbling property.

Key Words and Phrases: \(t\)-pebbling number, \(2t\)-pebbling property, shadow graph of path, Lourdusamy’s conjecture.

1 Introduction

Pebbling, one of the latest evolutions in graph theory proposed by Lakarias and Saks, has been the topic of vast investigation with significant observations. Having Chung [1] as the forerunner to familiarize pebbling into writings, many other authors too have developed this topic. Hulbert published a survey of graph pebbling [5].
Consider a connected graph with fixed number of pebbles distributed on its vertices. A pebbling move consists of the removal of two pebbles from a vertex and placement of one of those pebbles at an adjacent vertex. The $t$-pebbling number of a vertex $v$ in a graph $G$ is the smallest number $f_t(G, v)$ such that for every placement of $f_t(G, v)$ pebbles, it is possible to move $t$ pebbles to $v$ by a sequence of pebbling moves. Then the $t$-pebbling number of $G$ is the smallest number, $f_t(G)$ such that from any distribution of $f_t(G)$ pebbles, it is possible to move a pebble to any specified target vertex by a sequence of pebbling moves. Thus $f_t(G)$ is the maximum value of $f_t(G, v)$ over all vertices $v$.

The pebbling number is known for many simple graphs including paths, cycles, and trees, but it is not known for most graphs and is hard to compute for any given graph that does not fall into one of these classes. Therefore, it is an interesting question if there is information we can gain about the pebbling number of more complex graphs from the knowledge of the pebbling number of some graphs for which we know. In the first paper on graph pebbling [1] Chung proposed the following conjecture. The conjecture is perhaps the most compelling open question in graph pebbling known as Graham’s Conjecture.

**Conjecture 1.** (Graham [1]) For any connected graphs $G$ and $H$, $f(G \times H) \leq f(G).f(H)$.

Many articles [2], [3], [4] have given evidences supporting Conjecture 1. In proving Conjecture 1, two properties are used in the literature. They are $2$-pebbling property and odd $2$-pebbling property. In [8], Lourdusamy et al. proved that Conjecture 1 is true, when $G$ is a shadow graph of path and $H$ is a graph satisfying $2$-pebbling property. In [6], Lourdusamy had defined the $2t$-pebbling property of a graph.

**Definition 2.** [6] Given $t$-pebbling number of a graph $G$, let $p$ be the number of pebbles of $G$ and let $q$ be the number of vertices with at least one pebble. We say that $G$ satisfies the $2t$-pebbling property if it is possible to move $2t$-pebbles to any specified vertex whenever $p$ and $q$ satisfy the inequality $p + q > 2f_t(G)$.

Lourdusamy extended Graham’s conjecture (Conjecture 1) as follows:

**Conjecture 3.** (Lourdusamy [6]) For any connected graphs $G$ and $H$, $f_t(G \times H) \leq f(G).f_t(H)$.

This conjecture is also called the $t$-pebbling conjecture. Lourdusamy et al. [6], [7], [10], [11] proved that if $G$ is a fan graph, a wheel graph, a complete graph, a star graph, a complete $r$-partite graph and $H$ has the $2t$-pebbling property, then Conjecture 3 holds.

In this paper, show that Conjecture 3 is true, when $G$ is a shadow graph of path and $H$ is a graph satisfying $2$-pebbling property.

In Section 2, we give some theorems and definitions which are useful for the subsequent sections. In Section 3, we determine the $t$-pebbling number for the shadow graph of a path and we prove that the shadow graph of a path satisfies the $2t$-pebbling property. In Section 4, we show that the $t$-pebbling conjecture is
true, when $G$ is the shadow graph of a path and $H$ is any graph which satisfies the 2t-pebbling property.

2 Preliminaries

We begin with some theorems and definitions which are useful for the subsequent sections from [6], [8], [9] and [12].

**Theorem 4.** [6] Let $P_n$ be a path on $n$ vertices. Then $f_t(P_n) = t2^{n-1}$.

**Definition 5.** [12] The shadow graph $D_2(G)$ of a connected graph $G$ is constructed by taking two copies of $G$, say $G_1$ and $G_2$ and joining each vertex $u$ in $G_1$ to the neighbours of the corresponding vertex $v$ in $G_2$.

The shadow graph of a path is denoted by $D_2(P_n)$. Label the vertices in the first copy of the path by $x_1, x_2, \ldots, x_n$ and the vertices in the second copy of the path by $x_n+1, x_n+2, \ldots, x_{2n}$ starting from the left.

**Theorem 6.** [9] For the shadow graph of a path $P_n$, $f(D_2(P_n)) = 2^{n-1} + 2$.

**Theorem 7.** [9] The graph $D_2(P_n)$ satisfies the two-pebbling property.

**Theorem 8.** [8] Let $G$ be a graph which satisfies the 2t-pebbling property. Then $f(D_2(P_n) \times G) \leq (2^{n-1} + 2)f(G)$.

3 The t-Pebbling and 2t-pebbling property

In this section, we determine the $t$–pebbling number of the graph $D_2(P_n)$ and show that the graph $D_2(P_n)$ satisfies the 2t-pebbling property using Mathematical induction.

**Theorem 9.** For the shadow graph of a path $D_2(P_n)$, $f_t(D_2(P_n)) = t2^{n-1} + 2$, $n \geq 3$.

**Proof.** Placing $t2^{n-1}$ pebbles on $x_1$ and one pebble on each $x_n$ and $x_{n+1}$, we cannot move $t$ pebbles to $x_{2n}$. Thus $f_t(D_2(P_n)) \geq t2^{n-1} + 2$.

Now, consider the distribution of $t2^{n-1} + 2$ pebbles on the vertices of $D_2(P_n)$. Clearly, the result is true for $t = 1$ by Theorem 6. We assume the result is true for $2 \leq t' < t$. Let $v$ be the target vertex.

Case (i) Let $p(v) = 0$.

Clearly one pebble can be moved to $v$ at a cost of at most $2^{n-1}$ pebbles, since diameter of the graph $D_2(P_n)$ is $n-1$. Thus the remaining number of pebbles distributed on the vertices of $<D_2(P_n) - \{v\}>$ is at least $t2^{n-1} + 2 - 2^{n-1} = (t-1)2^{n-1} + 2$. Hence we can move $t - 1$ additional pebbles to $v$ by induction.

Case (ii) Let $p(v) = x$, where $1 \leq x \leq t - 1$. 

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Now the number of pebbles distributed on the vertices of $< D_2(p_n) - \{v\} >$ is $t2^{n-1} + 2 - x$. Since $t2^{n-1} + 2 - x \geq (t - x)2^{n-1} + 2$, we can move $t - x$ additional pebbles to $v$ by induction and hence we are done. \hfill \Box

**Theorem 10.** The graph $D_2(P_n)$ satisfies the 2t-pebbling property.

**Proof.** The proof is by induction on $t$. For $t = 1$, the result is true by Theorem 7. Assume the result is true for $2 \leq t' < t$. Consider the graph $D_2(P_n)$ with $2f_t(D_2(P_n)) - q + 1 = 2(t2^{n-1}) - q + 1$ pebbles on its vertices. Let $v$ be the target vertex. Then we consider the following cases.

Case (i) Let $p(v) = x$, where $1 \leq x \leq 2t - 1$.

Now the number of pebbles distributed on the vertices of the graph

$$D_2(P_n) - v \text{ is } 2(t2^{n-1}) - q + 1 - x \geq (2t - x)2^{n-1} = f(2t-x)(D_2(P_n)).$$

Thus we can move $2t - x$ additional pebbles to $v$ by Theorem 9.

Case (ii) Let $p(v) = 0$.

Clearly, we can move two pebbles to any vertex $v$ of $D_2(P_n)$ at a cost of at most $2^n$ pebbles and hence the remaining number of pebbles on the vertices of the graph $D_2(P_n)$ is at least

$$2(t2^{n-1}) - q + 1 - 2^n = 2((t - 1)2^{n-1}) - q + 1.$$ 

Thus we can move $2(t - 1)$ pebbles to the vertex $v$ by induction. \hfill \Box

4 Lourdusamy’s Conjecture

In this section, we show that the Conjecture 3 is true, when $G$ is the shadow graph of a path and $H$ is any graph having 2t–pebbling property.

**Definition 11.** [7] If $G = (V_G, E_G)$ and $H = (V_H, E_H)$ are two graphs, the direct product of $G$ and $H$ is the graph, $G \times H$, whose vertex set is the Cartesian product $V_{G \times H} = V_G \times V_H = \{(x, y) : x \in V_G, y \in V_H\}$ and whose edges are given by $E_{G \times H} = \{((x, y), (x', y')) : x = x' \text{ and } (y, y') \in E_H \text{ or } (x, x') \in E_G \text{ and } y = y'\}$.

We can depict $G \times H$ pictorially by drawing a copy of $H$ at every vertex of $G$ and connecting each vertex in one copy of $H$ to the corresponding vertex in an adjacent copy of $H$. We write $\{x\} \times H$ (respectively, $G \times \{y\}$) for the subgraph of vertices whose projection onto $V_G$ is the vertex $x$ (respectively, whose projection onto $V_H$ is $y$). If the vertices of $G$ are labeled $x_i$ then for any distribution of pebbles on $G \times H$, we write $p_i$ for the number of pebbles on $\{x_i\} \times H$ and $q_i$ for the number of occupied vertices of $\{x_i\} \times H$.

**Lemma 12.** (Transfer Lemma)[7] Let $(x_i, x_j)$ be an edge in $G$. Suppose that in $G \times H$, we have $p_i$ pebbles occupying $q_i$ vertices of $\{x_i\} \times H$, and $r_i$ of these
vertices have an odd number of pebbles. If \( r_i \leq k \leq p_i \) and if \( k \) and \( p_i \) have the same parity then \( k \) pebbles can be retained on \( \{a_i\} \times H \) while moving \( \frac{p_i-k}{2} \) pebbles onto \( \{a_j\} \times H \). If \( k \) and \( p_i \) have opposite parity we must leave \( k+1 \) pebbles on \( \{a_i\} \times H \), so we can move \( \frac{p_i-(k+1)}{2} \) pebbles onto \( \{a_j\} \times H \). In particular we can always move at least \( \frac{p_i-r_i}{2} \) pebbles onto \( \{a_j\} \times H \). In all these cases, the number of vertices of \( \{a_i\} \times H \) with an odd number of pebbles is unchanged by these moves.

Theorem 13. Let \( G \) be a graph which satisfies the \( 2t \)-pebbling property. Then \( f_t(D_2(P_n) \times G) \leq (2^{n-1}+2)f_t(G) \).

Proof. Let \( D_2(P_n) : x_1, x_2, \ldots, x_{2n} \). We prove this theorem by induction on \( n \). Since \( D_2(P_2) \) is isomorphic to \( C_4 \), by [4] the result is true for \( n = 2 \). Assume the result is true for \( 3 \leq n' < n \).

Let \( D \) be any distribution of \( (2^{n-1}+2)f_t(G) \) pebbles on the vertices of \( D_2(P_n) \times G \). Let \((x_1, y)\) be the target vertex, where \( y \) is in \( G \). By Transfer Lemma, we can transfer \( \frac{p_n-q_n}{2} \) pebbles from \( \{x_n\} \times G \) to \( \{x_{n-1}\} \times G \) and also \( \frac{p_{2n-q_{2n}}}{2} \) pebbles from \( \{x_{2n}\} \times G \) to \( \{x_{2n-1}\} \times G \). If

\[
p_1 + p_2 + \ldots + p_{n-1} + \frac{p_n-q_n}{2} + p_{n+1} + p_{n+2} + \ldots + p_{2n-1} + \frac{p_{2n}-q_{2n}}{2} \geq (2^{n-2}+2)f_t(G),
\]

then we can use the induction to put \( f_t(G) \) pebbles on \( \{x_1\} \times G \) and hence we reach the target.

Also, since \( G \) satisfies \( 2t \)-pebbling property, if

\[
\frac{p_n-q_n}{2} > 2^{n-3} f_t(G) \quad \text{and} \quad \frac{p_{2n}-q_{2n}}{2} > 2^{n-3} f_t(G),
\]

then we can put \( t2^{n-2} \) pebbles on \( (x_n, y) \) and \( t2^{n-2} \) pebbles on \( (x_{2n}, y) \). Thus we can move \( t2^{n-3} \) pebbles from \( (x_n, y) \) to \( (x_{n-1}, y) \) and \( t2^{n-3} \) pebbles from \( (x_{2n}, y) \) to \( (x_{n-1}, y) \). Then \( (x_{n-1}, y) \) contains at least \( t2^{n-2} \) pebbles and hence we can reach the target using the path \( P_{n-2} \).

Hence the only distributions from which we cannot reach the target \((x_1, y)\) satisfy the inequalities,

\[
p_1 + p_2 + \ldots + p_{n-1} + \frac{p_n-q_n}{2} + p_{n+1} + p_{n+2} + \ldots + p_{2n-1} + \frac{p_{2n}-q_{2n}}{2} < (2^{n-2}+2)f_t(G)
\]

and

\[
\frac{p_n-q_n}{2} + \frac{p_{2n}-q_{2n}}{2} \leq 2^{n-2}f_t(G).
\]

But adding these inequalities together we get,

\[
p_1 + p_2 + \ldots + p_{n-1} + p_n + p_{n+1} + p_{n+2} + \ldots + p_{2n-1} + p_{2n} < (2^{n-1}+2)f_t(G).
\]

Thus some configuration of pebbles from which we cannot pebble \( (x_1, y) \) must begin with fewer than \( (2^{n-1}+2)f_t(G) \) pebbles. By symmetry, we can pebble \( (x_n, y) \), \( (x_{n+1}, y) \) and \( (x_{2n}, y) \) using any configuration of \( (2^{n-1}+2)f_t(G) \) pebbles on \( f(D_2(P_n) \times G) \).

Let \((x_3, y)\) be the target vertex. Since \( G \) satisfies the \( 2t \)-pebbling property, if \( \frac{p_{n+1}-q_{n+1}}{2} > f_t(G) \), we can reach the target. Also if \( \frac{p_{n+1}-q_{n+1}}{2} > f_t(G) \), we can reach
the target. By Transfer Lemma, we can move $\frac{p_1-q_1}{2}$ pebbles to $(x_2, y)$ and $\frac{p_{n+1}-q_{n+1}}{2}$ pebbles to $(x_{n+2}, y)$. If

$$\frac{p_1-q_1}{2} + p_2 + p_3 + \ldots + p_n \frac{p_{n+1}-q_{n+1}}{2} + p_{n+2} + p_{n+3} + \ldots + p_{2n} \geq (2^{n-2} + 2)f_t(G),$$

then we can use induction to move $f_t(G)$ pebbles on $\{x_2\} \times G$.

Hence the only distributions from which we cannot pebble the target $(x_2, y)$ satisfy the inequalities

$$\frac{p_1-q_1}{2} \leq f_t(G) ; \frac{p_{n+1}-q_{n+1}}{2} \leq f_t(G) \text{ and } \frac{p_1-q_1}{2} + p_2 + p_3 + \ldots + \frac{p_{n+1}-q_{n+1}}{2} + p_{n+2} + p_{n+3} + \ldots + p_{2n} < (2^{n-2} + 2)f_t(G).$$

But adding these inequalities together we get,

$$p_1 + p_2 + \ldots + p_{n-1} + p_n + p_{n+1} + p_{n+2} + \ldots + p_{2n-1} + p_{2n} < (2^{n-1} + 2)f_t(G)$$

Thus some configuration of pebbles from which we cannot pebble $(x_1, y)$ must begin with fewer than $(2^{n-1} + 2)f_t(G)$ pebbles.

By symmetry, we can pebble $(x_{n-1}, y), (x_{n+2}, y)$ and $(x_{2n-1}, y)$ using any configuration of $(2^{n-1} + 2)f_t(G)$ pebbles on $D_2(P_n) \times G$.

Now let $(x_i, y)$ be the target vertex, where $i \in \{3, 4, \ldots , n-2, n+3, n+4, \ldots , 2n-2\}$. Then there are at least $(2^{n-3} + 2)f_t(G)$ pebbles on $< D_2(P_n) \times G - \{x_{n-1}\} \times G, \{x_n\} \times G, \{x_{2n-1}\} \times G, \{x_{2n}\} \times G \rangle$. Otherwise at least $(2^{n-3} + 2)f_t(G)$ pebbles are distributed on $< D_2(P_n) \times G - \{x_1\} \times G, \{x_2\} \times G, \{x_{n+1}\} \times G, \{x_{n+2}\} \times G \rangle$. Thus by induction we can reach the target vertex. \(\square\)

References


