Topological Approach to Multi-Fuzzy Approximation Spaces

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Abstract

Rough sets have been successfully used in addressing different aspects of uncertainty and vagueness. The rough set concept can be defined quite generally by means of topological operators, interior and closure, called approximations. This paper finds out that the collection of all lower approximation sets with respect to a reflexive and transitive multi-fuzzy relation forms a multi-fuzzy topology. The lower and upper approximation operators turn out to be the interior and closure operators respectively. The studies on the other way proves that there is a one-one correspondence between the set of all multi-fuzzy topologies and the set of all reflexive and transitive multi-fuzzy relations.

AMS Subject Classification: 68T37, 54A40, 94D05
Key Words and Phrases: Multi-fuzzy topology, multi-fuzzy approximation space, multi-fuzzy relation, reflexive, transitive.

1 Introduction

Rough set theory, introduced by Zdzislaw Pawlak in 1982 [1], has far reaching applications in numerous fields like data mining, machine learning, knowledge discovery, pattern recognition etc. The work by Dubois and Prade [2] was one among the earlier works in the direction of hybridization of rough set theory with fuzzy set theory. Since then, extensive research works have been carried out in integrating rough set theory with other theories related to soft technologies. Some examples can be found in [3, 4, 5, 6]. A lot of studies have been done in the direction of topological aspects of rough sets. Later, the studies are extended to the topological structures of hybrid set theories as in [7, 8].
Sabu Sebastian and T.V Ramakrishnan introduced the concept of multi-fuzzy sets as a generalization of fuzzy sets in [9] and multi-fuzzy rough sets are introduced in [10]. Studies on the topological properties of the newly formed hybrid structures give a more insight into their theory. The work in this paper is carried out using the definition of multi-fuzzy topology in [11]. In section 2, the essential ideas needed for the studies are reviewed. The connection between multi-fuzzy approximation spaces and multi-fuzzy topologies on a universe of discourse are discussed in the following sections.

2 Basic Concepts

Definition 1. [9] Let $W$ be a non-empty set, $\mathbb{N}$ the set of all natural numbers and \{${L}_i; i \in \mathbb{N}$\} a family of complete lattices. A multi-fuzzy set $P$ in $W$ is a set of ordered sequences

$$P = \{< a, \mu^1_P(a), \mu^2_P(a), ..., \mu^n_P(a), ... >; a \in W\}$$

where $\mu^i_P \in L_i^W$ (ie, $\mu^i_P : W \rightarrow L_i$) for $i \in \mathbb{N}$

Let $L_i = [0,1]$ for $i \in \mathbb{N}$, then the set of all multi-fuzzy sets in $W$, is denoted by $MF(W)$.

Definition 2. [12] Let $P$, $Q$ be any sets. Then a multi-fuzzy relation $R$ from $P$ to $Q$ is a multi-fuzzy subset of $P \times Q$

$$R = \{< (a,b), \mu^1_P(a,b), \mu^2_P(a,b), ..., >; a \in P, b \in Q\}$$

where $\mu^i_P : P \times Q \rightarrow L_i$ for $i \in \mathbb{N}$

Now the multi-fuzzy (MF) relation $R$ on $W$ is a multi-fuzzy subset of $W \times W$

$$R = \{< (a,b), \mu^1_P(a,b), \mu^2_P(a,b), ..., >; a,b \in W\}$$

where $\mu^i_P : W \times W \rightarrow L_i$ for $i \in \mathbb{N}$. If $L_i = [0,1]$, the set of all MF relations on $W$ is denoted by $MFR(W)$.

Some basic relations and operations on $MF(W)$ are defined as follows:

For every $P,Q \in MF(W)$,

a) $P \subseteq Q$ if and only if $\mu^i_P(a) \leq \mu^i_Q(a)$ for all $a \in W$ and for all $i \in \mathbb{N}$

b) $P = Q$ if and only if $\mu^i_P(a) = \mu^i_Q(a)$ for all $a \in W$ and for all $i \in \mathbb{N}$

c) $P \sqcup Q = \{< a, \mu^1_P(a) \lor \mu^1_Q(a), \mu^2_P(a) \lor \mu^2_Q(a), ..., >; a \in W\}$

d) $P \sqcap Q = \{< a, \mu^1_P(a) \land \mu^1_Q(a), \mu^2_P(a) \land \mu^2_Q(a), ..., >; a \in W\}$

The multi-fuzzy universe set is $1_W = \{< a, 1, 1, ..., >; a \in W\}$ and the multi-fuzzy empty set is $0_W = \{< a, 0, 0, ..., >; a \in W\}$.

For any $P \in MF(W)$, the complement of $P$ denoted by $\sim P$ is defined as, for $P = \{< a, \mu^1_P(a), \mu^2_P(a), ..., >; a \in W\}$,

$\sim P = \{< a, 1 - \mu^1_P(a), 1 - \mu^2_P(a), ..., >; a \in W\}$.

Definition 3. [10] A constant multi-fuzzy set, $\hat{\alpha}$ is defined as $\hat{\alpha} = \{< a, \alpha, \alpha, ..., >; a \in W\}$ with $\alpha \in [0,1]$. Thus the multi-fuzzy universe set is $1_W = \hat{1}$ and the multi-fuzzy empty set $0_W = \emptyset$.

Definition 4. [10] Let $R \in MFR(W \times W)$. Then $R$ is

a) reflexive if $\mu^i_R(a,a) = 1 \forall a \in W$ and $\forall i \in \mathbb{N}$
b) symmetric if $\mu_R^i(a, b) = \mu_R^i(b, a) \ \forall \ a, b \in W \ and \ \forall \ i \in N$

c) transitive if $\forall \ a, z \in W \ and \ \forall \ i \in N$

\[ \mu_R^i(a, z) \geq \text{Max}\{\text{Min}(\mu_R^i(a, b), \mu_R^i(b, z)) / b \in W\} \]

**Definition 5.** [11] A subset $\delta$ of $MF(W)$ is called a multi-fuzzy topology on $W$ if it satisfies the following conditions:

- $0_W, 1_W \in \delta$.
- $P \cap Q \in \delta$ for every $P, Q \in \delta$.
- $\forall \sigma \in \delta$ for every $\sigma \sqsubseteq \delta$, that is arbitrary union of multi-fuzzy sets in $\delta$ is in $\delta$.

**Definition 6.** [11] Let $(W, MF(W), \delta)$ be a multi-fuzzy topological space, $P \in MF(W)$.

1. *Interior of $P$* is the join of all the open subsets contained in $P$, denoted by $\text{int}(P)$, that is,

\[ \text{int}(P) = \lor\{v : v \sqsubseteq P, v \in \delta\} \]

2. *Closure of $P$* is the meet of all the closed subsets containing $P$, denoted by $\text{cl}(P)$, that is,

\[ \text{cl}(P) = \land\{u : u \sqsupseteq P, u^\prime \in \delta\} \]

The closure axioms that every closure operator $\text{cl}$ satisfy are the following:

1) $\text{cl}(P \cap (\overline{a_1}, \overline{a_2}, \ldots)) = \text{cl}(P) \cap (\overline{a_1}, \overline{a_2}, \ldots)$

2) For any multi-fuzzy set, $P \in MF(W)$, $\text{cl}(P) \subseteq P$

3) $\text{cl}(\text{cl}(P)) = \text{cl}(P) \ \forall P \in MF(W)$

4) $\text{cl}(P \sqcup Q) = \text{cl}(P) \sqcup \text{cl}(Q) \ \forall P, Q \in MF(W)$

The axioms of interior operator $\text{int}$ are given by

1) $\text{int}(P \sqcup (\overline{a_1}, \overline{a_2}, \ldots)) = \text{int}(P) \sqcup (\overline{a_1}, \overline{a_2}, \ldots)$

2) For any multi-fuzzy set, $P \in MF(W)$, $\text{int}(P) \subseteq P$

3) $\text{int}(\text{int}(P)) = \text{int}(P) \ \forall P \in MF(W)$.

4) $\text{int}(P \cap Q) = \text{int}(P) \cap \text{int}(Q) \ \forall P, Q \in MF(W)$.

**Definition 7.** [10] Let $W$ be a non-empty and finite universe of discourse and $R \in MFR(W \times W)$, the pair $(W, R)$ is called a multi-fuzzy approximation space. For $P \in MF(W)$, the family of all multi-fuzzy sets on $W$, the lower and upper approximations of $P$ with respect to (w.r.t) $(W, R)$ denoted by $\underline{R}(P)$ and $\overline{R}(P)$ are two multi-fuzzy sets and are, respectively, defined as follows:

\[ \underline{R}(P) = \{< a, \mu_{\underline{R}(P)}^1(a), \mu_{\underline{R}(P)}^2(a), \ldots, \mu_{\underline{R}(P)}^k(a), \ldots / a \in W\}, \]

\[ \overline{R}(P) = \{< a, \mu_{\overline{R}(P)}^1(a), \mu_{\overline{R}(P)}^2(a), \ldots, \mu_{\overline{R}(P)}^k(a), \ldots > / a \in W\}, \]
where
\[ \mu^i_{R(P)}(a) = \bigwedge \{ (1 - \mu^i_R(a,b), \mu^i_P(b)) / b \in W \} \]
\[ \mu^i_{R(P)}(a) = \bigvee \{ (\mu^i_R(a,b), \mu^i_P(b)) / b \in W \} \].

\( \forall i \in \mathbb{N} \) and \( \forall a \in W \)

\( R(P) \) and \( \overline{R}(P) \) are, respectively, called the lower and upper approximations of \( P \) w.r.t. \( (W, R) \). The pair \( (R(P), \overline{R}(P)) \) is called the MF rough set of \( P \) w.r.t. \( (W, R) \) and \( R, \overline{R} : MF(W) \to MF(W) \) are referred to as lower and upper multi-fuzzy rough approximation operators, respectively.

**Theorem 8.** [10] Let \( W \) be a non-empty and finite universe of discourse and \( R, R_1, R_2 \in MFR(W) \). Then the lower and upper approximation operators in definition 7 satisfy the following properties:

\( \forall P, Q \in MF(W) \),

(MFL1) \( \overline{R}(P) = \sim \overline{R}(\sim P) \)  
(MFL2) \( \overline{R}(P \sqcup (\alpha_1, \alpha_2, ...)) = \overline{R}(P) \sqcup (\alpha_1, \alpha_2, ...) \)  
(MFL3) \( \overline{R}(1_W) = 1_W \)  
(MFL4) \( \overline{R}(P \sqcap Q) = \overline{R}(P) \sqcap \overline{R}(Q) \)  
(MFL5) \( P \sqsubseteq Q \Rightarrow \overline{R}(P) \subseteq \overline{R}(Q) \)  
(MFL6) \( \overline{R}(P \sqcup Q) \sqsubseteq \overline{R}(P) \sqcup \overline{R}(Q) \)

(MFU1) \( \overline{R}(P) = \sim \overline{R}(\sim P) \)  
(MFU2) \( \overline{R}(P \sqcap (\alpha_1, \alpha_2, ...)) = \overline{R}(P) \sqcap (\alpha_1, \alpha_2, ...) \)  
(MFU3) \( \overline{R}(0_W) = 0_W \)  
(MFU4) \( \overline{R}(P \sqcup Q) = \overline{R}(P) \sqcup \overline{R}(Q) \)  
(MFU5) \( P \sqsubseteq Q \Rightarrow \overline{R}(P) \sqsubseteq \overline{R}(Q) \)  
(MFU6) \( \overline{R}(P \sqcup Q) \sqsubseteq \overline{R}(P) \sqcup \overline{R}(Q) \)

**Note:** The properties (MFL1) and (MFU1) shows that the multi-fuzzy rough approximation operators \( \overline{R} \) and \( \overline{R} \) are dual to each other.

**Theorem 9.** [10]

Let \( R \) be a MF relation on \( W \) and \( \overline{R} \) and \( \overline{R} \), the lower and upper MF rough approximation operators induced from \( (W, R) \). Then

(i) \( R \) is reflexive \( \iff (MFLR) \overline{R}(P) \subseteq P \ \forall P \in MF(W) \)
\( \iff (MFUR) P \subseteq \overline{R}(P) \ \forall P \in MF(W) \)

(ii) \( R \) is transitive \( \iff (MFLT) \overline{R}(P) \subseteq \overline{R}(\overline{R}(P)) \ \forall P \in MF(W) \)
\( \iff (MFUT) \overline{R}(\overline{R}(P)) \subseteq \overline{R}(P) \ \forall P \in MF(W) \)

3 Multi-fuzzy Topological Spaces generated by Multi-fuzzy Approximation Spaces

Note that, in this section we assume \( R \) as a reflexive and transitive multi-fuzzy relation.

**Lemma 1.** Let \( R \) be a reflexive and transitive multi-fuzzy relation on \( W \). For all \( A_j \in MF(W), \ j \in I \) (I is an index set),

\[ \overline{R}(\bigcup_{j \in I} \overline{R}(A_j)) = \bigcup_{j \in I} \overline{R}(A_j) \]  

(1)

**Proof.** The proof follows from the reflexivity and transitivity of \( R \). □
Theorem 10. Let \( R \) be a reflexive and transitive multi-fuzzy relation on \( W \). The set of all lower approximations of multi-fuzzy sets on \( W \) forms a multi-fuzzy topology, i.e.,

\[
T_R = \{ R(P) / P \in MF(W) \}
\]
is a multi-fuzzy topology on \( W \).

Proof. a) To prove \( 0_W, 1_W \in T_R \).
Since \( R(1_W) = 1_W \) by (Varma and John, 2014), \( 1_W \in T_R \).
Also, since \( R \) is reflexive, \( R(0_W) \subseteq 0_W \) and hence \( 0_W \in T_R \).

b) If \( P,Q \in T_R \), there exists \( A_1,B_1 \in MF(W) \) such that \( P = R(A_1), Q = R(B_1) \).
\( P \cap Q = R(A_1) \cap R(B_1) = R(A_1 \cap B_1) \).
Thus \( P \cap Q \in T_R \) for \( P,Q \in T_R \).

c) For all \( A_j \in T_R \),
\[
\bigcup_{j \in I} A_j = \bigcup_{j \in I} R(B_j) = R\left(\bigcup_{j \in I} R(B_j)\right) \quad \text{by lemma 1}
\]
\[
\Rightarrow \bigcup_{j \in I} A_j \in T_R
\]
Thus the approximation space \((W,R)\) generates the topology \( T_R \). \( \square \)

Theorem 11. Let \( T_R \) be the multi-fuzzy topology as defined in theorem10. Then \( \forall MF(W), \)

1. \( R(P) = \text{int}(P) = \sqcup\{ R(Q) / R(Q) \subseteq P \} \)
2. \( \overline{R}(P) = \text{cl}(P) = \cap\{ \sim R(Q) / \sim R(Q) \ni P \} \)

Proof. 1. Note that, \( R(P) \subseteq P \) and \( R(R(P)) \subseteq R(P) \) by (Varma and John, 2014)
\[
\mu^i_{\text{int}(P)}(a) = \bigvee_{R(Q) \subseteq P} \mu^i_{R(Q)}(a) \\
\geq \mu^i_{R(P)}(a) \quad (2)
\]
By the reflexivity and transitivity of \( R \) and by Theorem 2 in (Varma and John, 2014) we have,
\( \forall R(Q) \subseteq P \)
\[
\mu^i_{R(Q)}(a) \leq \mu^i_{R(P)}(a) \quad \forall a \in W, \forall i \in \mathbb{N} \quad (3)
\]
\[
\mu^i_{\text{int}(P)}(a) = \bigvee_{R(Q) \subseteq P} \mu^i_{R(Q)}(a) \\
\leq \mu^i_{R(P)}(a) \quad (4)
\]
From (3.2) and (3.4),
\[
R(P) = \text{int}(P) \quad (5)
\]
2. The duality of the operators $\overline{R}$ and $R$ and 5 give the proof.

4 Multi-fuzzy Approximation Spaces generated from Multi-fuzzy Topologies

In this section, we examines the conditions by which a multi-fuzzy topological space can be associated with a multi-fuzzy approximation space.

Let $f_1(W), f_2(W), f_3(W), \ldots$ denote the families of membership functions in multi-fuzzy sets such that for $P, Q \in MF(W)$, $\mu^i_P, \mu^i_Q \in f_1(W)$, $\mu^d_P, \mu^d_Q \in f_2(W)$, ..., also we define, $\mu^i_P \sqsubseteq \mu^i_Q$ iff $\mu^i_P(a) \leq \mu^i_Q(a) \ \forall a \in W$ and $\mu^i_P = \mu^i_Q$ iff $\mu^i_P(a) = \mu^i_Q(a) \ \forall a \in W$.

**Definition 12.** Let $H : MF(W) \rightarrow MF(W)$ be a multi-fuzzy operator defined by $H = (H_{\mu_1}, H_{\mu_2}, \ldots, H_{\mu_k}, \ldots)$ where $H_{\mu_i} : f_i(W) \rightarrow f_i(W)$ for $i = 1, 2, 3, \ldots$

**Theorem 13.** Let $cl, int : MF(W) \rightarrow MF(W)$ be the closure and interior operators of the multi-fuzzy topological space $(W, MF(W), \delta)$. Then there exists a multi-fuzzy relation $R_\delta$ on $W$ such that $int(P) = R_\delta(P)$ and $cl(P) = \overline{R_\delta}(P) \ \forall P \in MF(W)$ iff $int$ satisfies axioms (IA1) and (IA2), or equivalently, $cl$ satisfies axioms (CA1) and (CA2):

$a) \ \forall P, Q \in MF(W), \alpha_i \in [0, 1], \text{ for } i = 1, 2, \ldots$

IA1) $int(P \sqcup (\alpha_1, \alpha_2, \ldots) = int(P) \sqcup (\alpha_1, \alpha_2, \ldots)$

IA2) $int(P \sqcap Q) = int(P) \sqcap int(Q)$

IA3) $int_{\mu_i}(\mu_P^i \sqcup \hat{\alpha}_i) = int_{\mu_i}(\mu_P^i) \sqcup \hat{\alpha}_i$

IA4) $int_{\mu_i}(\mu_P^i \sqcap \hat{\alpha}_i) = int_{\mu_i}(\mu_P^i) \sqcap \hat{\alpha}_i$

CA1) $cl(P \sqcap (\alpha_1, \alpha_2, \ldots) = cl(P) \sqcap (\alpha_1, \alpha_2, \ldots)$

CA2) $cl(P \sqcup Q) = cl(P) \sqcup cl(Q)$

CA3) $cl_{\mu_i}(\mu_P^i \sqcup \hat{\alpha}_i) = cl_{\mu_i}(\mu_P^i) \sqcup \hat{\alpha}_i$

CA4) $cl_{\mu_i}(\mu_P^i \sqcap \hat{\alpha}_i) = cl_{\mu_i}(\mu_P^i) \sqcap \hat{\alpha}_i$

Proof. Assume that there exists a $MF$ relation $R$ on $W$ such that $int(P) = R_\delta(P)$ and $cl(P) = \overline{R_\delta}(P), \ \forall P \in MF(W)$. We have to prove that the interior operator, $int$ satisfies (IA1) and (IA2) or the closure operator $cl$ satisfies (CA1) and (CA2).

The proof follows from (MFL2) and (MFU2) in theorem 8.

Now conversely suppose that the operator $cl$ satisfies axioms (CA1) and (CA2). Let us define a relation

$R_\delta = \{ (a, b) \mid \mu_{R_\delta}^i(a, b), \mu_{R_\delta}^d(a, b), \ldots, \mu_{R_\delta}^k(a, b) > \langle a, b \rangle \in W \times W \}$

on $W$ by $cl$ as follows:

$$\mu_{R_\delta}^i(a, b) = cl_{\mu_i}(\mu_{1_b}^i(a, b), \ \forall (a, b) \in W \times W$$

For any $P \in MF(W)$,

$$\mu_P^i = \bigsqcup_{b \in W} \{ (\mu_P^i \cap (\mu_{1_b}^i)(b)) \}$$
Then for any \( a \in W \), according to (CA1) and (CA2), we have
\[
\mu^i_{R_\delta(P)}(a) = \bigvee_{b \in W} [(\mu^i_{R_\delta}(a, b) \land \mu^i_P(b))]
\]
\[
= \bigvee_{b \in W} [cl_{\mu}(\mu^i_{1\mu})(a) \land \mu^i_P(b)]
\]
\[
= \bigvee_{b \in W} [cl_{\mu}((\mu^i_{1\mu}) \cap \mu^i_P(b))(a)]
\]
\[
= \bigvee_{b \in W} cl_{\mu}([\mu^i_{1\mu}(b) \cap \mu^i_P(b)])(a)
\]
\[
= cl_{\mu}([\cup_{b \in W} ([\mu^i_{1\mu}(b) \cap \mu^i_P(b)])](a)
\]
\[
= cl_{\mu}(\mu^i_P)(a)
\]
\[
= \mu^i_{cl(P)}(a)
\]
Thus \( cl(P) = \overline{R_\delta}(P) \)

Similarly \( int(P) = R_\delta(P) \) can be proved. Now it remains to prove that the multi-fuzzy relation \( R_\delta \) is reflexive and transitive.

By theorem 9, it is enough to prove that \( P \subseteq R_\delta(P) \) and \( \overline{R_\delta}(\overline{R_\delta}(P)) \subseteq \overline{R_\delta}(P) \ \forall P \in MFS(W) \). Since \( cl(P) = \overline{R_\delta}(P) \), by the definition of closure, \( P \subseteq cl(P) = \overline{R_\delta}(P) \).

Also, \( \overline{R_\delta}(\overline{R_\delta}(P)) = cl(cl(P)) = P \). Hence \( R_\delta \) is reflexive and transitive.

**Theorem 14.** Let \( R \) be the set of all reflexive and transitive multi-fuzzy relations on \( W \) and \( M \) be the set of all multi-fuzzy topologies on \( W \) satisfying the axioms (IA1), (IA2), (CA1) and (CA2). Then there exists a one-to-one correspondence between \( R \) and \( M \).

**Proof.** Let \( R \in R \). Let \( T_R \) be defined as in 10 and \( R_{T_R} \) as in 6. Since \( R \) is a reflexive and transitive \( MF \) relation, by 11, \( \overline{R} = \overline{int(T_R)} \) and \( \overline{R} = \overline{cl(T_R)} \). From 6,
\[
\mu^i_{R_{T_R}}(a, b) = cl_{\mu}(\mu^i_{1\mu})(a)
\]
\[
= \mu^i_{\overline{R}(1\mu)}(a), \ \forall (a, b) \in W \times W
\]
\[
= \mu^i_{\overline{R}(1\mu)}(a)
\]
\[
= \text{Max}\{\text{Min}(\mu^i_R(a, b'), \mu^i_{1\mu}(b'))/b' \in W\}
\]
\[
= \text{Min}(\mu^i_R(a, b), \mu^i_{1\mu}(b))
\]
\[
= \mu^i_R(a, b).
\]
Thus \( R_{T_R} = R \).

In a similar manner, from 11 and 13, we get, for \( T \in M \), \( T_{R_T} = T \) where \( R_T \) is defined as in 6 and \( T_{R_T} \) as in 10. From these relations, we can define one-to-one correspondences between \( R \) and \( M \). Hence the result.

**5 Conclusion**

The paper tried to find the interplay between theory of multi-fuzzy sets and the corresponding approximation spaces from a topological approach. It is proved that
when the multi-fuzzy relation involved is reflexive and transitive, there is an inherent multi-fuzzy topology. On the other hand, we proved that a multi-fuzzy relation which is reflexive and transitive can be associated with each multi-fuzzy topology.

References


