Application of Differential Equation in Stability Analysis of Dependent Viscosity of thermohaline Convection in Ferromagnetic Fluid in Densely Packed Porous Medium

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Abstract

The study of ferrothermohaline convection with multi component fluids has wide range of application in heat and mass transfer. This work deals with the theoretical investigation of the effect magnetic field dependent viscosity on ferrothermohaline convection heated and salted from below in an anisotropic porous medium subjected to a transverse uniform magnetic field. Eigen value problem solved by Brinkman model and normal mode technique used for the vertical anisotropic porous medium. The stability analysis has been made for different parameters. A small thermal perturbation is applied to the basic state and linear stability analysis is used for which normal mode technique is applied. It is found that the present work has been carried out for that the oscillatory as well as stationary modes with graphical representation.

Key Words: Thermohaline convection, soret effect, porous medium, salinity rayleigh number, darcy model.
1. Introduction

Ferrofluid is a liquid, which becomes strongly magnetized in the presence of magnetic field. There are many fascinating materials, which have been attracting scientists and researchers for their extraordinary physical properties and technical usage. Ferrofluid is one of such smart materials not available in nature freely, but is to be synthesized by different processes. The three components required to prepare Ferrofluids are magnetic particles of colloidal size, carrier liquid and stabilizer (surfactant). They are stable suspensions of colloidal single domain ferromagnetic particles of the order of 10 nm in suitable non-magnetic carrier liquid they are used in shock absorbers, vacuum seals, stepper motors and they are used as transformer coolants. Ferro fluids are single magnetic domain, two phase three-component fluids (Rosensweig, 1985) where the core represents the single domain, core and carrier fluids represent the two phases, and core, surfactant and carrier fluids represent the three components. Finlayson studied the effect of ferroconvection of a single-component fluid. He explained the concept of thermo mechanical interaction in Ferrofluids (Finlayson, 1970). Das and Gupta (1979) studied the stability effect of rotation on setting up of convective instability in Ferrofluids. Sekar et al (1997) studied the ferrothermohaline convection. The effect of anisotropy on ferroconvective instability saturating a porous medium of high permeability has been analysed (Sekar et al) (1996). Vaidyanathan et al (2005) have been analyzed conductive and convective heat transfer in Ferrofluids. Degroot and Mazur (1962) studied the thermodynamic properties of multicomponent fluids. Sunil et al (2004, 2005, and 2006) studied the effect of magnetic field dependent viscosity on ferroconvection in dusty ferrofluids with, without porous medium and rotating porous medium. Sunil et al (2005a, 2005b) studied the presence and absence of porous medium on the effect of magnetic field dependent viscosity on thermosolutal convection in ferromagnetic fluid. Also, Sunil et al (2004) showed the thermal convection in ferromagnetic fluids in a porous medium with effect magnetic field dependent viscosity. Sunil et al (2008) analysed a nonlinear study of stability analysis for thermal convection in a ferromagnetic fluid with magnetic field dependent viscosity. Sekar et al (2000, 2002) analyzed ferrothermohaline convection in which a horizontal layer of an incompressible ferromagnetic fluid of thickness ‘d’ in the transverse magnetic field heated from below and salted from above is considered. Sekar et al (2013) they studied and analyzed the condition for the onset of convection by considering the soret effect. The present paper is analyze the effect of densely packed soret-driven ferrothermohaline convection in a porous medium.

2. Mathematical formulation

An infinitely spread thin layer of Boussinesq Ferrofluids of thickness ‘d’ in the presence of transverse applied magnetic field heated and salted from below is considered. The temperature and salinity at the bottom surface and at the upper
surface $z = \pm \frac{d}{2}$ are $T_0 \pm \frac{\Delta T}{2}$ and $S_0 \pm \frac{\Delta S}{2}$ respectively. Further, the system is assumed to have anisotropy along the vertical direction, which is taken as the $z$-axis. The governing mathematical equations are as follows.

The fluid viscosity is assumed magnetic dependent in the form (Vaidyanathan et al., 2002)

$$
\mu = \mu_i (1 + \sigma \mathbf{B})
$$

The continuity equation for an incompressible Boussinesq fluid

$$
\nabla \cdot \mathbf{q} = 0
$$

The momentum equation for Darcy model which viscosity $\mu$ is

$$
\rho_0 \frac{D \mathbf{q}}{Dt} = -\nabla p + \rho \mathbf{g} + \nabla \cdot (\mathbf{H} \mathbf{B}) - \frac{\mu_i (1 + \sigma \mathbf{B})}{k} \mathbf{q}
$$

The temperature equation for an incompressible fluid that modified Fourier law and as given by Finlayson (1970) is

$$
\left[ \rho_i C_v \frac{dT}{dt} - \mu_i T \frac{\partial^2 \mathbf{M}}{\partial t^2}, \frac{dT}{dt} \right] + \mu_i T \left( \frac{\partial \mathbf{M}}{\partial t}, \frac{\partial \mathbf{H}}{\partial t} \right) = K_1 \nabla^2 T + \phi
$$

The mass flux equation is

$$
\rho_0 \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{S} = K_S \nabla^2 \mathbf{S} + S_T \nabla^2 T
$$

The Maxwell’s equations for non-conducting fluids are

$$
\nabla \cdot \mathbf{B} = 0, \nabla \times \mathbf{H} = 0
$$

Further, the magnetic field, magnetization and magnetic induction are related by

$$
\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})
$$

The density equation of state for Boussinesq magnetic fluid is

$$
\rho = \rho_0 \left[ 1 - \alpha_i (T - T_0) + \alpha_s (S - S_0) \right]
$$

One can assume that the magnetization depends on the magnetic field and temperature, so that

$$
\mathbf{M} = \frac{\mathbf{H}}{H} \mathbf{M}(H, T, S)
$$

The magnetic equation of state is linearized about the magnetic field $H_0$ and the average temperature $T$ of the lower and upper surface of the layer is

$$
\mathbf{M} = M_0 + \chi (H - H_0) - k(T - T_0) + K_2 (S - S_0)
$$

The basic state is assumed quiescent state. The basic state quantities are obtained by substituting velocity of quiescent state in equations (2)-(5). The basic state quantities obtained are as follows

$$
q_b = 0, T_b = T_0 - \beta_T z, \quad S_b = S_0 - \beta_S z, \quad \rho(z) = \rho_0 \left[ 1 + \alpha_i \beta_T z - \alpha_s \beta_S z \right], \quad p = p_b(z)
$$

$$
H_b(z) = \frac{k}{1 + \chi} \left[ \frac{K_2 \beta_T z}{1 + \chi} + \frac{K_1 \beta_S z}{1 + \chi} \right]
$$
\[ M_b (z) = k \left[ M_0 + \frac{K \beta z}{1 + \chi} \frac{K_2 \beta z}{(1 + \chi)} \right] \]  \hspace{1cm} (13)

Where \( k \) is the unit vector in the vertical direction. In the case of Finalyson (1970) and Sekar et al (2013). The perturbation equation can be obtained by imposing small thermal perturbation on all the dynamical quantities

The modified Navier-Stoke equation can be obtained on linearization as

\[ \rho_0 \frac{\partial \mathbf{u}}{\partial t} = \frac{\partial \mathbf{p}}{\partial x} + \mu_0 (M_0 + H_0) \frac{\partial H'_1}{\partial z} - \frac{\mu_1}{k_1} \mathbf{u} \]  \hspace{1cm} (14)

\[ \rho_0 \frac{\partial \mathbf{v}}{\partial t} = \frac{\partial \mathbf{p}}{\partial y} + \mu_0 (M_0 + H_0) \frac{\partial H'_1}{\partial z} - \frac{\mu_1}{k_1} \mathbf{v} \]  \hspace{1cm} (15)

\[ \rho_0 \frac{\partial \mathbf{w}}{\partial t} = \frac{\partial \mathbf{p}}{\partial z} + \mu_0 (M_0 + H_0) \frac{\partial H'_1}{\partial z} - \frac{\mu_1}{k_1} \mathbf{w} - \mu_0 \mathbf{K} \mathbf{K} \mathbf{H}_T S_0 \left( 1 + \alpha \right) - \mu_0 \mathbf{K} \mathbf{K} \mathbf{g} \mathbf{m} \mathbf{T} - \mu_0 \mathbf{K} \mathbf{K} \mathbf{H}_T S_0 - \mu_0 \mathbf{K} \mathbf{K} \mathbf{H}_T S_0 \left( 1 + \alpha \right) \]  \hspace{1cm} (16)

Where \( k_1 \) and \( k_2 \) are the permeability of horizontal and vertical direction, the vertical component of momentum equation can be calculated as

\[ \rho_0 \frac{\partial (\nabla w)}{\partial t} = \rho_0 g \alpha_1 \nabla_{z} \theta - \rho_0 g \alpha_1 \nabla_{z} S^0 + \left( \mu_0 K_2 \beta_0 \right) (1 + \alpha) \left[ K (1 - S_T) \nabla_{z} \theta - (1 + \chi) \frac{\partial}{\partial z} (\nabla_{y} \phi) \right] \]  \hspace{1cm} (17)

\[ + \left( \frac{\mu_0 K_2}{1 + \chi} \left[ \beta_0 (1 - S_T) \nabla_{z} \theta - \beta_0 \nabla_{y} S^0 \right] - \frac{\mu_0 K_2}{k_1} \frac{\partial^2 w}{\partial z^2} - \frac{\mu_0}{k_2} \nabla_{z} \theta - \frac{\mu_0 \partial \mu_0 (M_0 + H_0)}{k_1} (\nabla_{y} w) \right) \]  \hspace{1cm} (18)

where \( \rho_0 C = \rho_0 C_{v, w} + \rho_0 K H_0 \)

\[ \text{The salinity equation is} \]  \hspace{1cm} (19)

\[ \frac{\partial S}{\partial t} + \beta \mathbf{w} = K_T (\nabla^2 S) + S_T (\nabla^2 \theta) \]

Using the normal mode techniques for solve the stability problem

\[ f (x, y, z, t) = f (z, t) \exp i(k_x x + k_y y) \]  \hspace{1cm} (20)

where \( k_0 \) is the wave number is given by

\[ k_0 = \sqrt{(k_x^2 + k_y^2)} \]  \hspace{1cm} (21)

Following normal mode technique, using the equations (20) and (21) the vertical component of (16)-(19) can be written as

\[ \rho_0 \frac{\partial}{\partial t} \left( \frac{\partial^2}{\partial z^2} - k_0^2 \right) w = \frac{\mu_0 K_2}{1 + \chi} \frac{\partial^2 \phi}{\partial z^2} - k_0^2 (1 - S_T) \nabla_{z} \theta - \rho_0 g \alpha_1 k_0^2 \theta + \rho_0 g \alpha_1 k_0^2 S \]
The boundary condition for stress free boundaries is

\[
\rho_0 C_{v,H} \frac{\partial \theta}{\partial t} - \mu_0 T_0 \frac{\partial \phi}{\partial t} = K_1 \left( \frac{\partial^2}{\partial z^2} - k_0^2 \right) \theta
\]

(23)

\[
\frac{\partial S}{\partial t} + \beta_w w = K_s \left( \frac{\partial^2}{\partial z^2} - k_0^2 \right) S + S_T \left( \frac{\partial^2}{\partial z^2} - k_0^2 \right) \theta
\]

(24)

\[
(1 + \chi) \frac{\partial^2 \phi}{\partial z^2} - \left( 1 + \frac{M_0}{H_0} \right) k_0^2 - K \frac{\partial \theta}{\partial z} + K_2 \frac{\partial S}{\partial z} + S_T K \frac{\partial \theta}{\partial z} = 0
\]

(25)

The following non-dimensional numbers have been used

\[
w^* = \frac{wd}{v}, \quad t^* = \frac{vt}{d^2}, \quad T^* = \left( \frac{K_m aR \frac{1}{2}}{\rho_0 C_{v,H} K \beta v d} \right) \theta^*, \quad \phi^* = \left( \frac{(1 + \chi)K_m aR \frac{1}{2}}{\rho_0 C_{v,H} K \beta v d^2} \right) \phi^*, \quad \frac{z^*}{d} = \frac{z}{d}
\]

(26)

Expressing the above equations in non-dimensional form by introducing the various non-dimensional term and parameters, the equations (22)-(25) then gives the non-dimensional equations

\[
\frac{\partial}{\partial t^*} \left( D^2 - a^2 \right) w^* = \left[ M_D \phi^* - (1 + M_1(1 - S_T))T^* \right] a^{1/2} + \left( D^2 - a^2 \right) w^*
\]

(27)

\[
+ aR^2 M_1 M_4 D \phi^* - aR^2 M_1 M (1 - S_T) T^* + aR_5 \left[ 1 + M_4 + M_1 M^{-1} \right] S^*
\]

\[
- \frac{1}{k_1} D^2 w^* + \frac{a^2}{k_2} w^* - a^2 M_4 \delta^* \left( D^2 - a^2 \right) - \frac{1}{k_2} \right) w^*
\]

(27)

\[
p_1 \frac{\partial T^*}{\partial t} - M_2 p_1 \frac{\partial}{\partial t^*} \left( D \phi^* \right) = \left( D^2 - a^2 \right) T^* + (1 - M_2 M_5) aR^2 w^*
\]

(28)

\[
p_1 \frac{\partial S^*}{\partial t} = \tau \left( D^2 - a^2 \right) S^* - aR_5 v^2 M_s w^* + S_T M_4 M_6 \left( \frac{R}{R_s} \right) D^2 - a^2) T^*
\]

(29)

and

\[
D^2 \phi^* - a^2 M_4 \phi^* - DT^* + M_5 M_6^{-1} \left( R / R_s \right)^2 DS^* = 0
\]

(30)

3. Analysis of Solution at Free Boundaries

The boundary condition for stress free boundaries is
\[
\begin{align*}
\phi^* &= \frac{E}{\pi} e^{\tau^*} \sin(\pi \tau^*) \\
\text{(32)}
\end{align*}
\]

Where A, B, C and D are constants using the equations (31) in (27)-(29) and dropping the asterisks for convenience leads to the following set of three linear, homogeneous algebraic equations in the constants A, B and they are

\[
\begin{align*}
\left[ (\pi^2 + a^2) \sigma + \left( \frac{\pi^2}{k_1} + \frac{a^2}{k_2} \right) + \left( \pi^2 + a^2 \right) + a^2 M_3 \delta \left( \pi^2 + a^2 + \frac{1}{k_2} \right) \right] A \\
+ a^2 R \left[ 1 + M_1 M_3 (1 - S_1) + M (1 - S_1) \right] B \\
+ a R^2 (1 + M_4 + M_5 M_7^{-1}) C + \\
a^2 R^2 M_4 (1 + M_3) D
\end{align*}
\]

\[
\begin{align*}
\text{(33)}
\end{align*}
\]

\[
\begin{align*}
ar^2 \left[ 1 - M_2 - M_7 M_3 \right] A - \left[ \left( \pi^2 + a^2 \right) + pr \sigma \right] B + M_2 pr \sigma D = 0
\end{align*}
\]

\[
\begin{align*}
ar^2 \left[ S_3 M_3 e^{-i \left( \frac{2 \pi}{R} \right)} \left( \pi^2 + a^2 \right) B + M (\pi^2 + a^2) + \sigma \right] C = 0
\end{align*}
\]

\[
\begin{align*}
R^2 \left[ \pi^2 (1 - S_1) B + R \left( \pi^2 + a^2 M_7 \right) C + R^2 \left( \pi^2 + a^2 M_7 \right) C = 0
\end{align*}
\]

Where \( \varepsilon \) is a non dimensional parameter, which is the ratio of vertical to horizontal plane permeability, governing the anisotropy. Solving the homogeneous equations 32-35 for non trivial solutions, the determinant of coefficient of A, B, C and D are made to vanish. This leads to the algebraic equation given by

\[
\begin{align*}
-U \sigma^2 + V \sigma^2 + W \sigma + X = 0
\end{align*}
\]

Where

\[
\begin{align*}
U &= - (\pi^2 + a^2) (\pi^2 + a^2 M_3) pr \left( \pi^2 + a^2 \right) + \pi^2 + a^2 \pi^2 (1 - S_1) \\
V &= \left( \pi^2 + a^2 \right) M_3 (1 + M_3) pr \left( 1 + \pi^2 + a^2 M_3 \right) A + M (1 + \pi^2 + a^2 M_3) + \\
W &= \left( \pi^2 + a^2 \right) M_3 \left( \pi^2 + a^2 M_3 \right) + \left( \pi^2 + a^2 \right) \pi^2 M_3 + \left( \pi^2 + a^2 M_3 \right) A + \\
X &= \left( \pi^2 + a^2 \right) \left( \pi^2 + a^2 M_3 \right) \pi^2 + a^2 \left( \pi^2 + a^2 M_3 \right) + a^2 R (1 - M_3) \left( \pi^2 + (1 + M_1) a^2 M_3 \right)
\end{align*}
\]

Making \( \sigma = 0 \) in equation (37) condition for steady convection is obtained and substitution \( \sigma = \sigma_1 \), gives the non-zero condition for oscillatory convection. From equation (35), the Rayleigh number for over stability can be easily obtained as Sekar et al., (2013)

\[
\begin{align*}
R_C^m = \left[ \left( \sigma_1 c_1 + c_2 \right) a \sigma_2 c_1 \left( \sigma_1 c_1 + c_2 \sigma_1 \right) + \frac{c_2}{c_1} \left( \sigma_1 c_1 + c_2 \sigma_1 \right) - c_2 c_1 \sigma_1 \right] / \left( c_1 c_2 \sigma_1 \right)
\end{align*}
\]

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\[
\begin{align*}
R_C = \left[ \left( \sigma c_1 + c_2 \right) a \sigma_2 c_1 \left( \sigma_1 c_1 + c_2 \sigma_1 \right) + \frac{c_2}{c_1} \left( \sigma_1 c_1 + c_2 \sigma_1 \right) - c_2 c_1 \sigma_1 \right] / \left( c_1 c_2 \sigma_1 \right)
\end{align*}
\]

For physical consideration of thermal convection, \( R_C \) has to be real and hence
the equation (38) becomes

\[ R_C = \left\{ \frac{c_4(a^2c_1 + c_4) + \sigma c_1(c_1^2c_2 + \sigma c_6)}{c_4^2 + c_5^2 + \delta^2} \right\} \]  

(39)

Where

\[ c_1 = p_r(\pi^2 + a^2)(1 + \tau)p_r^2A + T_2 \]
\[ c_2 = (\pi^2 + a^2)(\pi^2 + a^2M_s)\pi^2 - (\pi^2 + a^2)p_rM_s \pi^2(1 - S_T) \]
\[ c_3 = -\left(\pi^2 + a^2\right) + p_r\left\{ (\pi^2 + a^2)(\pi^2 + a^2M_s)(1 + \tau) \right\} + a^2R_s(\pi^2 + a^2M_s)p_r \]
\[ c_4 = a^2(\pi^2 + a^2) \left\{ \tau(\pi^2 + a^2)(\pi^2 + a^2M_s)(1 - M_2 - M_4)(1 + \tau) \right\} \]
\[ c_5 = (\pi^2 + a^2) p_r(M_1(1 + M_s)(1 - S_T) - M_s) + (1 + M_2(1 + M_s) - M_s)^2 \]
\[ c_6 = (a^2 + \pi^2)(a^2M_s + \pi^2) \tau \]

Figure 1: The effect of magnetic field on the variation of \( N_C \) \( V_S \) \( \delta \) for different values \( S_T, R_s, S, \tau, \varepsilon \) and \( k_1 = 0.1 \)

Figure 2: The effect of magnetic field on the variation of \( N_C \) \( V_S \) \( \delta \) for different values \( S_T, R_s, S, \tau, \varepsilon \) and \( k_1 = 0.01 \)

Figure 3: The effect of magnetic field on the variation of \( N_C \) \( V_S \) \( \delta \) for different values \( S_T, R_s = 500, S_T = 0.002, \tau = 0.11, K_1 = 0.01, \) and \( \varepsilon = 3.1 \)
Figure 4: The effect of magnetic field on the variation of $N_C V_S M_3$ for different values of $\delta$, $R_S=500$, $S_T=0.002$, $\tau=0.11$, $K_1=0.01$, and $\varepsilon=3.1$

Figure 5: The effect of magnetic field on the variation of $N_C V_S M_3$ for different values of $\delta$, $R_S=-500$, $S_T=-0.002$, $\tau=0.03$, $K_1=0.01$, and $\varepsilon=0.3$

Figure 6: The effect of magnetic field on the variation of $N_C V_S M_3$ for different values of $\delta$, $R_S=100$, $S_T=0.001$, $\tau=0.09$, $K_1=0.01$, and $\varepsilon=2.1$

Figure 7: The effect of magnetic field on the variation of $N_C V_S \tau$ for different values of $\delta=0.01$, $R_S=500$, $S_T=0.002$, $K_1=0.1$, and $\varepsilon=3.1$
The role of effective of on thermo convective instability of ferromagnetic fluids heated from below and salted from above considered with magnetic field dependent viscosity for different values of magnetic field dependent viscosity \(\delta\) =0.01to 0.09 (Vaidyanathan et al., 2002). The stability analysis is applied Brinkman model and small perturbation method. The Soret parameter \(S_T\) is assumed to take the values -0.002 to 0.002, the Rayleigh number \(R_S\) is varied from -500 to 500, the magnetization parameter \(M_3\) is allowed and takes from 1 to 9, the ratio of the mass transport to heat transport values \((\tau)\) is varied from 0.03 to 0.11, the Prandtl number \(P_r\) is taken as 0.01, the permeability parameter \(k_1\) considered as 0.01 and 0.1. The Magnetization parameter \(M_1\) to be 1000 (Sekar et al., 2013). For these fluids \(M_2\) will have negligible value.

From the figures 1, 2 and 3 shows that the variation of \(N_c\) with \(\delta\) for different values of \(S_T\), \(k_1\), \(\tau\). It is obvious from these figure that the coefficient of Magnetic field dependent viscosity \(\delta\) has stabilizing effect in the system. The stabilizing effect of \(\delta\) is much pronounced. In the Figures 4, 5 and 6 represents the variation of \(N_c\) versus \(M_3\) is analyzed, for different values of \(\delta\) is increased from 0.01 to 0.09, \(\varepsilon\) is increased from 0.3 to 3.1 and \(k_1\) increased from 0.01 to 0.1. It is increased that the system that the system gets the destabilizing effect as \(N_c\) decreases. It is also seen from the figures the critical thermal magnetic Rayleigh number \(N_c\) reduced due to \(M_3\). Because magnetization relaxes extra energy which adds up to thermal energy to promote convection. Figure 6 and 7 shows the variation \(N_c\) versus the ratio of mass transport to heat transport \(\tau\) for various \(\delta\) and for \(k=0.1\). When \(\tau\) increases from 0.03 to 0.11, there is decrease in \(N_c\) promoting instability when \(\delta=0.01\) whereas increasing the values of \(\delta\) from 0.01 to 0.03, \(N_c\) gets decreasing the value. It is observed that the system gets stabilizing effect.

References


