Alternative rings with some Lie and Jordan product identities in the center

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Abstract: Quadri and et al. studied the commutativity of associative rings satisfying the identities (i) \([x, y]^2 = [x^2, y^2]\) and (ii) \((x \circ y)^2 = x^2 \circ y^2\) for all \(x, y \in R\). Here \([x, y] = xy - yx\) is the Lie product and \((x \circ y) = xy + yx\) is the Jordan product. Giri and Rakhunde proved the commutativity of associative rings not necessarily with unity satisfying these identities in the center by using Herstein’s theorem. RamAwtar, Giri, Rakhunde and Modi proved some results on commutativity of associative rings with certain identities in the center using Herstein theorem. In this paper without using Herstein theorem in nonassociative rings, we prove some results on commutativity of an alternative ring \(R\) satisfying any one of the identities:

(i) \([x, y]^2 - [x^2, y^2] \in U\),
(ii) \([(xy)^2 - xy, z] \in U\) or \([(xy)^2 - yx, z] \in U\) or
(iii) \([x^2y^2 - xy, z] \in U\) or \([x^2y^2 - yx, z] \in U\)
(iv) \((x \circ y)^2 - (x^2 \circ y^2) \in U\)
(v) \((x \circ y)^2 - (x \circ y) \in U\) for all \(x, y \in R\) and for fixed \(z \in R\).

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Introduction: An alternative ring \(R\) is a ring in which \((x, x, y) = 0\) and \((y, x, x) = 0\) for all \(x, y \in R\). These equations are known as the left and right alternative laws respectively. The center \(U\) of \(R\) is defined as \(U = \{u \in R / [u, R] = 0\}\). It is also called as commutative center. A ring \(R\) is of characteristic \(\neq n\) if \(nx = 0\) implies \(x = 0\) for all \(x \in R\) and \(n\) a natural number.

First we prove the following Lemmas:

Lemma 1: Let \(R\) be an alternative ring with \([x, [x, y]] = 0\). Then \(2[x, y]^2 = [x, [x, y^2]]\).

Proof: We have
\[
[x, [x, y^2]] - 2[x, y]^2 = [x, xy^2 - y^2x] - 2(xy - yx)^2
= x(xy^2 - y^2x) - (xy^2 - y^2x)x - 2(xy^2 - y^2x)x - 2(xy^2 - y^2x)^2 + 2(xy)(yx) + 2(yx)(xy)
= x^2y^2 - x(y^2x) - x(y^2x) + y^2x^2 - 2(xy^2 - y^2x)^2 + 2x(y^2x) + 2y(x^2y)
= x^2y^2 + y^2x^2 - 2(xy)^2 - 2(yx)^2 + 2y(x^2y)
= (x^2y^2 - 2(xy)^2 + (yx)^2) + (y^2x^2 - 2(xy)^2 + y(x^2y))
\]
\[
\begin{align*}
&= (x^2y - 2(xy)x + yx^2)y + y(yx^2 - 2x(yx) + x^2y) \\
&= [x, [x, y]]y + y[x, [x, y]] = 0.
\end{align*}
\]

Therefore \(2[x, y]^2 = [x, [x, y]].\)

**Lemma 2**: Let \(R\) be a prime alternative ring satisfying the condition \([x, y]^2 - [x^2, y^2] \in U\) for all \(x, y\) in \(R\). Then \(R\) has no nonzero nilpotent elements.

**Proof**: By hypothesis \([x, y]^2 - [x^2, y^2] \in U\)

i.e., \((xy-xy)^2 - (x^2y^2 - y^2x^2) \in U.\)

Now we replace \(x\) with \(x + y\). Then

\[\begin{align*}
((x + y)y - y(x + y))^2 - (x + y)^2y^2 + y^2(x + y)^2 = 0,
\end{align*}\]

or

\[\begin{align*}
(x - yx)^2 - x^2y^2 - (xy)y^2 - (yx)^2y^2 + y^4 + y^2x^2 + y^2(xy) + y^2(yx) + y^4 = 0.
\end{align*}\]

Using (1), we get \(y^2(xy) - (yx)y^2 + y^2(yx) - (xy)y^2 \in U.\)

Thus for \(y \in R\), we have

\[\begin{align*}
(y^2(xy) - (yx)y^2 + y^2(yx) - (xy)y^2)y = y(y^2(xy) - (yx)y^2 + y^2(yx) - (xy)y^2).
\end{align*}\]

Without loss of generality let \(0 \neq x \in R\) and \(x^2 = 0.\)

Now by replacing \(y\) with \(yx\) in (3), we get

\[\begin{align*}
&= (yx)(y(xy) - (yx)(yx) + (yx)((yx)(yx) - (x(xy))(yx)^2)\)\)

\[\begin{align*}
&= (yx)(y(xy) - (yx)(yx) + (yx)((yx)(yx) - (x(xy))(yx)^2)\)
\end{align*}\]

i.e., \(yxyxyxyx - yxyxyxyx + yxyxyxyx - yxyxyxyx\)

\[\begin{align*}
&= yxyxyxyx - yxyxyxyx + yxyxyxyx - yxyxyxyx.
\end{align*}\]

Now using the fact \(x^2 = 0\) in the above equation, we obtain \((xy)^4x = 0\), that is \((xy)^5 = 0\) for all \(y \in R\).

By Lemma 1.1 of [4] it follows that \(xR\) is a nonzero right ideal of \(R\) in which \(z^5 = 0\) where \(z \in xR\). But \((xy)^5 = 0\) implies \(xR = 0\), since \(x^2 = 0\). Then \(xRx = 0\). Hence \(x = 0\), by primeness of \(R\).

**Theorem 1**: Let \(R\) be a prime alternative ring of char. \(\neq 2\) satisfying \([x, y]^2 - [x^2, y^2] \in U\) for all \(x, y\) in \(R\). Then \(R\) is commutative.

**Proof**: By hypothesis \([x, y]^2 - [x^2, y^2] \in U\)

i.e., \((xy - xy)^2 - (x^2y^2 - y^2x^2) \in U.\)

Now by replacing \(x\) with \(x + y\) in (4) and simplifying, we get

\[\begin{align*}
y^2(xy) - (yx)y^2 + y^2(yx) - (xy)y^2 \in U.
\end{align*}\]

Therefore

\[\begin{align*}
(y^2(xy) - (yx)y^2 + y^2(yx) - (xy)y^2)y = y(y^2(xy) - (yx)y^2 + y^2(yx) - (xy)y^2),
\end{align*}\]

for all \(x, y\) in \(R\). Now we get \(y^4x - 2y^2(xy^2) + xy^4(5)\)

This can be rewritten as \([y^2, [y^2, x]] = 0.\)
By replacing $x$ with $x^2$ in (6), we get $[y^2, [y^2, x^2]] = 0$. \hspace{1cm} (7)
From Lemma 1, we have $2[y^2, x]^2 = 0$.

Now Lemma 2 yields $2[y^2, x] = 0$.

Since $R$ is of char. $\neq 2$, we have $[x, y^2] = 0$. Hence $[x, [x, y^2]] = 0$.

Using Lemma 1, we have $2[x, y]^2 = 0$.

Since $R$ is of char. $\neq 2$, we get $[x, y]^2 = 0$.

From Lemma 2, this implies $[x, y] = 0$.

i.e., $xy = yx$. Hence $R$ is commutative.

**Theorem 2**: Let $R$ be a prime alternative ring with char. $\neq 2$ satisfying (i) $[(xy)^2 − xy, z] \in U$
or (ii) $[(xy)^2 − yx, z] \in U$ for all $x, y$ in $R$ and for fixed $z$ in $R$. Then $R$ is commutative.

**Proof**: (i) By hypothesis $[(xy)^2 − xy, z] \in U$. \hspace{1cm} (8)
Now by replacing $x$ with $x + y$ in (8), we get

$$[(xy + y^2)^2 − xy − y^2, z] \in U$$
or

$$[(xy)^2 + (xy)y^2 + y^2(xy)) + y^4 − xy − y^2, z] \in U.$$ \hspace{1cm} (9)

Using (8) in (9), we get

$$[(xy)y^2 + y^2(xy) + y^4 − y^2, z] \in U.$$ \hspace{1cm} (10)

By substituting $x = y$ in (8), we get

$$[y^4 − y^2, z] \in U.$$ \hspace{1cm} (11)

Using (11) in (10), we obtain

$$[(xy)y^2 + y^2(xy), z] \in U.$$ \hspace{1cm} (12)

Again by replacing $x$ with $x + y$ in (12), we get

$$[((x + y)y)y^2 + y^2((x + y)y), z] \in U$$
or

$$[(xy)y^2 + y^2(xy) + 2y^4, z] \in U.$$ \hspace{1cm} (13)

Now using (12), we obtain $[2y^4, z] \in U$. Since $R$ is of char. $\neq 2$, we get

$$[y^4, z] \in U.$$ \hspace{1cm} (14)

By using (11) in (13), we have $[y^2, z] \in U$.

We replace $y$ with $xy$ in (14). Then $[(xy)^2, z] \in U$. \hspace{1cm} (15)

Using (8) and (15), we obtain $[xy, z] \in U$. \hspace{1cm} (16)

Now we replace $z$ with $yx$ in (16). Then $[xy, yx] \in U$ (or) $(xy)^2x − (yx)^2y \in U$.

First we shall prove that $U \neq (0)$.

Let us suppose that $U = (0)$, i.e., $(xy)^2x = (yx)^2y$ \hspace{1cm} (17)

for all $x, y$ in $R$.

By replacing $y$ with $y + y^2$ in (17), we obtain

$$(x(y^2 + y^4 + 2y^3)x = (yx^2 + y^2x^2)(y + y^2).$$
i.e., $(xy)^2x + (xy)^4x + 2(xy)^3x = (yx)^2y + (yx)^4y + (y^2x^2)y + (y^2x^2)y^2$
i.e., $2(xy)^3x = (y^2x^2)y + (yx)^2y^2$. \hspace{1cm} (18)

We replace $y$ by $y + y^3$ in (17). Then $2(xy)^4x = (y^3x^2)y + (yx)^3y^3$ or
\[
2(y^2x^2)y^2 = y^2((yx^2)y) + ((yx^2)y)y^2 = y^2((xy^2)x) + ((xy^2)x)y^2
\]

We write this as \((y^2x^2)y^2 - y^2((xy^2)x) = ((xy^2)x)y^2 - \cdots\)

Let \(I_y^2\) be the inner derivation by \(y^2\), i.e., \(x \rightarrow xy^2 - y^2x\)

and \(I_y^3\) be the inner derivation by \(y^3\). Then (19) becomes \(I_y^3 I_y^2(x) = 0\).

Thus the product of these derivations is again a derivation.

Then by the Theorem 1 in [5] we can conclude that either \(y^2\) or \(y^3\) in \(U\), i.e., \(y^2\) or \(y^3\) is zero.

If \(y^3 = 0\), then (18) becomes \((y^2x^2)y + (yx^2)y^2 = 0\).

Substituting \(x + y\) for \(x\), we get \((y^2(x + y)^2)y + (yx(x + y)^2)y^2 = 0\).

i.e., \((y^2x^2 + y^2(xy) + y^2(xy) + y^4)y + (yx^2 + y(xy) + y(xy) + y^3)y^2 = 0\).

Then we get \(2(y^2x)y^2 = 0\) or \((y^2x)y^2 = 0\) or \((y^2R)y^2 = 0\). Then \(y^2 = 0\).

Thus if \(U = (0)\), then \(y^2 = 0\) for every \(y\) in \(R\).

Then \(0 = (x + y)^2x = (xy)x\) or \(xRx = 0\).

Then \(x = 0\) or \(R = 0\), a contradiction. Therefore \(U \neq (0)\).

Taking \(\lambda \neq 0\) in \(U\) and let \(x = x + \lambda\) in \((xy^2)x - (yx^2)y\) in \(U\), we get

\[\lambda(xy^2 - 2(yx)y + y^2x)\] in \(U\).

Since \(R\) is prime, we must have

\[xy^2 - 2(yx)y + y^2x\] in \(U\).

(20)

If \(\lambda a\) is in \(U\), then \(\lambda(ab - ba) = 0 = \lambda(ab - ba)\).

Then \(R\lambda(ab - ba) = 0 = \lambda R(ab - ba)\) and since \(\lambda \neq 0\),

we have \(ab - ba = 0\), i.e., \(a\) is in \(U\).

In (20), we let \(x = xy\) and get

\[(xy^2 - 2(yx)y + y^2x)y\] in \(U\), then \(y\) is in \(U\),

Unless \(xy^2 - 2(yx)y + y^2x = 0\). So if \(y\) is not in \(U\),

\(xy^2 - 2(yx)y + y^2x = 0\), for every \(x\) in \(R\) and \(y\) is in \(U\), then

\(xy^2 - 2(yx)y + y^2x\) is still zero.

Therefore, \(xy^2 + y^2x = 2(yx)y\),

(21) for every \(x, y\) in \(R\).

Since \(R\) is of char. \(\neq 2\), then \(R\) is commutative by Lemma [3].

(ii) By hypothesis \([(xy)^2 - yx, z] \in U\).

(22)
Now we replace $x$ with $x + y$ in (22) and using (22), we obtain
\[(xy)y^2 + y^2(xy) + y^4 - y^2, z \in U.\] (23)
By replacing $x = y$ in (22) and using this in (23), we get
\[(xy)y^2 + y^2(xy), z \in U.\] (24)
Now applying the same argument as in the Theorem 2(i), we conclude that $R$ is commutative.

**Theorem 3:** Let $R$ be a prime alternative ring with char. $\neq 2$ satisfying (i) $[x^2y^2 – xy, z] \in U$ or (ii) $[x^2y^2 – xy, z] \in U$ for all $x, y$ in $R$ and for fixed $z$ in $R$. Then $R$ is commutative.

**Proof:**
(i) By hypothesis $[x^2y^2 – xy, z] \in U$. (25)
We replace $x$ with $x + y$ in (25). Then
\[([x + y]y^2 – (x + y)y, z] \in U\]
or
\[[x^2y^2 + (xy)y^2 + (yx)y^2 – xy + y^4 – y^2, z] \in U.\] (26)
Using (25) in (26), we get $([xy)y^2 + (yx)y^2 + y^4 – y^2, z] \in U.$ (27)
By replacing $x$ with $y$ in (25), we obtain $[y^4 – y^2, z] \in U.$ (28)
Using (28) in (27), we get $([xy)y^2 + (yx)y^2, z] \in U.$ (29)
Now by replacing $x = x + y$ in (29), we have $([xy)y^2 + (yx)y^2 + 2y^4, z] \in U.$
Using (29) in above and applying char. $\neq 2$, we get $[y^4, z] \in U.$ (30)
From (28) and (30), we have $[y^2, z] \in U.$ (31)
By replacing $y = x + y$ in (31) and using (31), we get
\[[xy + yx, z] \in U.\] (32)
Now by replacing $z = yx$ in (32), we get $[xy, yx] \in U$ or $(xy^2)x – (yx^2)y \in U.$
Now applying the same argument as in the Theorem 2(i), we conclude that $R$ is commutative.
(ii) By hypothesis
\[[x^2y^2 – xy, z] \in U.\] (33)
We replace $x$ with $x + y$ and use (33). Then
\[[[xy)y^2 + (yx)y^2 + y^4 – y^2, z] \in U.\]
Now applying the same argument as in Theorem 3(i), we conclude that $R$ is commutative.

**Theorem 4:** Let $R$ be an alternative ring with char. $\neq 2$ satisfying $(x \circ y)^2 – (x \circ y^2) \in U$ for all $x, y$ in $R$. Then $R$ is commutative.

**Proof:** By hypothesis $(x \circ y)^2 – (x \circ y^2) \in U$
or
\[(xy + yx)^2 – (x^2y^2 + y^2x^2) \in U.\] (34)
Now by replacing $x$ with $x + y$ in (34), we get
\[((x+y)y + y(x+y))^2 – ((x+y)y^2 + y^2(x+y)^2) \in U\]
i.e., $(xy + yx + 2y^2)^2 – (x^2 + xy + yx + y^2)y^2 – y^2(x^2 + xy + yx + y^2) \in U.$
On simplification, we obtain
\[(xy + yx)^2 – x^2y^2 – y^2x^2 + y^2(xy + yx) + (xy + yx)y^2 + 2y^4 \in U.\] (35)
Using (34) in (35), we get

\[ y^2(xy + yx) + (xy + yx)y^2 + 2y^4 \in U. \quad (36) \]

This equation asserts

\[ (y^2(xy) + y^2(yx) + (xy)y^2 + 2y^4)y = y(y^2(xy) + y^2(yx) + (xy)y^2 + (yx)y^2 + 2y^4), \]

for all \( x, y \) in \( R \).

By simplifying, we obtain \( xy^4 - y^4x = 0 \).

i.e., \([x, y^4] = 0\). Hence \([x, [x, y^4]] = 0\).

By Lemma 1, we get \( 2[x, y^2]^2 = 0 \).

Since \( R \) is of char. \( \neq 2 \), we get \([x, y^2]^2 = 0\).

By using Lemma 2, we get \([x, y^2] = 0\).

Hence \([x, [x, y^2]] = 0\).

Again by using Lemma 1, we get \( 2[x, y]^2 = 0 \).

Since \( R \) is of char. \( \neq 2 \) and using Lemma 2, we get \([x, y] = 0\)

i.e., \( xy = yx \). Hence \( R \) is commutative.

**Theorem 5:** Let \( R \) be an alternative ring of char. \( \neq 2 \) satisfying \((x \circ y)^2 - (x \circ y) \in U\) for all \( x, y \) in \( R \). Then \( R \) is commutative.

**Proof:** By hypothesis \((x \circ y)^2 - (x \circ y) \in U\)

or \((xy + yx)^2 - (xy + yx) \in U\). \hspace{1cm} (37)

Now by substituting \( x = x + y \) in (37), we get

\[ ((x + y)y + y(x + y))^2 - (x + y)y - y(x + y) \in U \]

or \((xy + yx + 2y^2)^2 - xy - yx - 2y^2 \in U\).

On simplification, we obtain

\[ (xy + yx)^2 - xy - yx - 2y^2(xy + yx) + 2(xy + yx)y^2 + 2y^2 \in U. \quad (38) \]

Using (37) in (38), we get \( 2y^2(xy + yx) + 2(xy + yx)y^2 + 2y^2 \in U\). \hspace{1cm} (39)

Since \( R \) is of char. \( \neq 2 \), we get

\[ y^2(xy) + y^2(yx) + (xy)y^2 + (yx)y^2 + y^2 \in U. \quad (40) \]

This equation asserts

\[ (y^2(xy) + y^2(yx) + (xy)y^2 + (yx)y^2 + y^2)y = y(y^2(xy) + y^2(yx) + (xy)y^2 + (yx)y^2 + y^2), \]for \( x, y \) in \( R \).

Now applying the same argument as in the Theorem 1, we conclude that \( R \) is commutative.

**Reference:**
