Labeling Techniques of Some Special Graphs

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Abstract

A graph G with q edges is said to be harmonious if there is a injective f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label f(x)+f(y)(mod q), the resulting edge labels are distinct. A function f is called harmonious labelling of graph G if f: v → {0,1,2,...,q-1} is injective and the induced function f*: E→{0,1,2,..,q} defined as f*(e=uv)=(f(u)+f(v)(mod q)) is bijective. In this paper, we compute graceful labeling, Harmonious labeling, Zumkeller labeling of Ladder graph, banana tree and Firecracker graph.

Key Words: Graceful labeling, zumkeller labeling, Harmonious labeling.
1. Introduction

Alex Rosa introduced graceful labeling in 1967. A labeling of the graph is assigning a nature values to the vertices of the graph in some way that induced edge labels according to certain pattern. A particular topic of interest was on labelling of graphs specifically, on harmoniously labeling graphs. The work of Tanna (2013) involved reiterations of proofs, as well as, supplementary examples to an earlier work of Graham and Sloane (1980) concerning harmonious labelling of certain classes of graphs. Zumkeller number published in Sloane’s sequences of integers A083207 a sequence of integer \(n\) with the property that the positive factor of \(n\) can be partitioned into two disjoint parts so that the sums of the two parts are equal.

2. Preliminaries

A. Graceful Labeling [1, 9]

**Definition 2.1**

Let \(G = (V, E)\) be a simple graph with order \(n\) and size \(m\). Let \(V = \{v_1, v_2, \ldots, v_n\}\) and \(E = \{e_1, e_2, \ldots, e_m\}\). Let each node \(v_i\) be labeled with distinct non-negative integer \(x_i\). A map \(f\) defined on the Cartesian product of the set of labels of vertices to the set of labels of edges is called graceful if \(f(x_i, x_j) = |x_i - y_j|\) such that each \(|x_i - y_j|\) is distinct and vertices from it \(e_m\) and the graph is \(G\) is called graceful graph.

**Theorem 2.1.1** [5]: If \(n > 4\), the complete graph \(K_n\) is not graceful.

**Theorem 2.1.2** [7]: If \(G\) is an Eulerian graph with \(m\) edges such that \(m \equiv 1\) or \(2(\text{mod } 4)\), then \(G\) cannot be labeled gracefully.

**Theorem 2.1.3** [6]: Let \(T\) and \(S\) be two graceful trees under the valuations \(f_1\) and \(f_2\) respectively and Let \(|T| = m\) and \(|S| = n\).

**Theorem 2.1.4**[10]: All caterpillars are graceful.

B. Harmonious Labeling

**Definition 2.1.2[4]**

Let \(G\) be a graph with \(q\) edges. A function \(f\) is called harmonious labeling of graph \(G\) if \(f: v \rightarrow \{0, 1, 2, \ldots, q-1\}\) is injective and the induced function \(f^*: E \rightarrow \{0, 1, 2, \ldots, q\}\) defined as \(f^*(e = uv) = (f(u) + f(v) \pmod{q})\) is bijective. A graph which admits harmonious labeling is called harmonious graph.

**Theorem 2.2.1**[8]: The cycle \(C_n\) (\(n \geq 3\)) is harmonious if and only if \(n\) is odd.

**Theorem 2.2.2**[4]: The path \(P_n\) is a prime harmonious labeling.

**Theorem 2.2.3**[4]: The cycle \(C_n\) is prime harmonious labeling.
C. Zumkeller Labeling
Definition 2.3.1[2]

Let \( G = (V, E) \) be a graph. An injection function \( f: V \rightarrow \mathbb{N} \) is said to be a Zumkeller labelling of the graph \( G \) if the induced function \( f': E \rightarrow \mathbb{N} \) defined as \( f'(xy) = f(x)f(y) \) is a Zumkeller number for all \( xy \in E, x, y \in V \).

Theorem 2.3.1[2]: The complete bipartite graph \( K_{m,n} \) is a Zumkeller graph.

Theorem 2.3.2[3]: Every full binary tree admits a Zumkeller labeling.

Theorem 2.3.3[2]: The helm graph \( H_n \) admits a Zumkeller labeling, when \( n \equiv 0 \pmod{2} \).

Theorem 2.3.4[2]: The graph \( C_n^* \) admits a Zumkeller labeling, when \( n \equiv 0 \pmod{2} \).

D. Ladder Graph
Definition 2.4.1

The ladder graph \( L_n \) (n\( \geq \)2) is the product graph \( \mathbb{P}_2 \times \mathbb{P}_n \) which contains 2n vertices and 3n-2 edges.

Theorem 2.4.1 [9]: The ladder graph \( L_n \) with m-pendant edges \( L_n, mk_1 \) is odd graceful.

Theorem 2.4.2[1]: The subdivision of ladder graph \( L_n \) with m-pendant edges \( s(L_n), mk_1 \) is odd graceful.

Theorem 2.4.3[1]: All the subdivision of triangular snakes (\( \Delta_k - snake \)) with m-pendant edges \( s(\Delta_k - snake), mk_1 \) are odd graceful.

3. Labeling of Ladder Graph

Graceful Labeling of Ladder Graph
For n=2,

Let \( G = \{V, E, f\} \) be a graph with \( V = \{0,1,2,5\}, E = \{1,2,3,4\} \) and \( f \) be defined as \( f(0,1) = 1; f(0,2) = 2; f(2,5) = 3; f(1,5) = 4 \).

For n=3,

Let \( G = \{V, E, f\} \) be a graph with \( V = \{0,1,4,5,7,10\}, E = \{1,2,3,4,5,6,7\} \) and \( f \) be defined as \( f(0,1) = 1; f(7,5) = 2; f(1,4) = 3; f(1,5) = 4; f(5,10) = 5; f(10,4) = 6 \);
\( f(0,7) = 7. \)

For \( n=4 \)

Let \( G = \{ V, E, f \} \) be a graph with \( V = \{ 0,1,4,9,10,8,12,3 \} \), \( E = \{ 1,2,3,4,5,6,7,8,9,10 \} \) and \( f \) be defined as \( f(0,1) = 1; f(10,8) = 2; f(1,4) = 3; f(8,12) = 4; f(4,9) = 5; f(9,3) = 6; f(1,8) = 7; f(4,12) = 8; f(12,3) = 9; f(0,10) = 10. \)

Zumkeller Labeling of Ladder Graph

For \( n=2 \)

24, 20, 40, 48 be a Zumkeller numbers.

For \( n = 3 \)

88, 12, 42, 220, 40, 30, 56 be a Zumkeller numbers.
For \( n = 4 \)

\[
\begin{array}{c}
4 & (28) \\
12 & (96) \\
30 & (20) \\
7 & (56) \\
20 & (60) \\
(40) \\
40 & (30) \\
30 & (60) \\
0 & (160) \\
12 & (48) \\
28 & (12) \\
30 & (30) \\
96 & (96) \\
56 & (20) \\
60 & (160) \\
(48) \\
& (20) \\
\end{array}
\]

28, 48, 12, 30, 96, 56, 20, 160, 60, 40 be a Zumkeller number.

4. Labeling of Banana Tree

Definition 4.1 (Banana tree)

Banana tree as defined is graph obtained by connecting one leaf of each of \( n \) copies of a \( K \)-star graph with single root vertex that is distinct from all the stars.

A. Graceful Labeling of \( B_{2,4} \)

Let \( G \) be a graph with 9 vertices and 8 edges and \( f \) defined as \( f(0,1) = 1; f(1,3) = 2; f(3,6) = 3; f(13,9) = 4; f(8,13) = 5; f(13,7) = 6; f(3,10) = 7; f(0,8) = 8. \)

\[
\begin{array}{c}
9 & (4) \\
15 & (6) \\
8 & (3) \\
(5) & (3) \\
(1) & (2) \\
0 & (1) \\
\end{array}
\]

\( B_{2,5} \)

Let \( G \) be a graph with 11 vertices and 10 edges and \( f \) defined as \( f(0,1) = 1; f(1,3) = 2; f(3,6) = 3; f(3,7) = 4; f(3,8) = 5; f(10,4) = 6; f(4,11) = 7; f(4,12) = 8; f(4,13) = 9; f(0,10) = 10. \)
**B\(_{3,4}\)**

Let G be a graph of 13 Vertices and 12 Edges and f is defined as:

\[
\begin{align*}
&f(0,1) = 1; \\
&f(1,3) = 2; \\
&f(3,6) = 3; \\
&f(3,7) = 4; \\
&f(0,5) = 5; \\
&f(5,11) = 6; \\
&f(11,18) = 7; \\
&f(11,19) = 8; \\
&f(22,31) = 9; \\
&f(12,22) = 10; \\
&f(22,33) = 11; \\
&f(0,12) = 12.
\end{align*}
\]

**B\(_{3,5}\)**

Let G be a graph of 16 Vertices and 15 edges and f defined as:

\[
\begin{align*}
&f(0,1) = 1; \\
&f(5,3) = 2; \\
&f(5,8) = 3; \\
&f(1,5) = 4; \\
&f(5,10) = 5; \\
&f(0,6) = 6; \\
&f(6,13) = 7; \\
&f(13,21) = 8; \\
&f(13,22) = 9; \\
&f(13,23) = 10; \\
&f(0,11) = 11; \\
&f(24,12) = 12; \\
&f(11,24) = 13; \\
&f(24,38) = 14; \\
&f(24,39) = 15.
\end{align*}
\]
Let G be a graph of 17 Vertices and 16 edges and f is defined as $f(0,1) = 1$; $f(1,3) = 2$; $f(3,6) = 3$; $f(3,7) = 4$; $f(0,5) = 5$; $f(5,11) = 6$; $f(11,18) = 7$; $f(11,19) = 8$; $f(0,9) = 9$; $f(9,19) = 10$; $f(19,30) = 11$; $f(19,31) = 12$; $f(0,13) = 13$; $f(13,27) = 14$; $f(27,42) = 15$, $f(27,43) = 16$.

Let G be a graph of 21 Vertices (20) Edges (20) and f defined as $f(0,1) = 1$; $f(1,3) = 2$; $f(3,6) = 3$; $f(3,7) = 4$; $f(0,5) = 5$; $f(0,6) = 6$; $f(6,13) = 7$; $f(13,21) = 8$; $f(13,22) = 9$; $f(13,23) = 10$; $f(0,11) = 11$; $f(24,36) = 12$; $f(11,24) = 13$; $f(24,38) = 14$;
f(24,39) = 15, f(0,16) = 16, f(16,33) = 17, f(33,51) = 18, f(33,52) = 19, f(33,53) = 20.

Definition 1.5 (Fire Cracker Graph)

An Fire cracker graph (n,k)-fire cracker is a graph obtained by the concatenation of n K-stars by linking one leaf.

B. Fire Cracker Graph of Graceful Labeling

$F_{2,4}$

Let $G$ be a graph of 8 vertices and 7 edges and $f$ is defined as $f(0,1) = 1$; $f(0,2) = 2$; $f(2,5) = 3$; $f(2,6) = 4$; $f(8,13) = 5$; $f(8,14) = 6$; $f(1,8) = 7$.

$F_{2,5}$

Let $G$ be a graph of 10 vertices and 9 edges and $f$ is defined as $f(0,1) = 1$; $f(1,3) = 2$; $f(3,6) = 3$; $f(3,7) = 4$; $f(3,8) = 5$; $f(9,15) = 6$; $f(9,16) = 7$; $f(9,17) = 8$; $f(0,9) = 9$. 

$B_{4,5}$
5. Conclusion

In this paper Graceful labelling, Harmonious labelling and Zumkeller labelling of ladder graph, Banana tree and fire cracker graphs are computed.

References


