Finite Topology BASED Image Analysis

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Abstract

The structure of images is explained using topological Cellular Complex (CC), which is the lone finite sets topology, and is discussed for encoding images. Under this topology, while defining the linked subsets and their borders, no inconsistencies or ironies arise. In the topological sense, image segmentation is breaking a CC into Cell-Blocks (CB). For any Segmented Images, an accurate and compact data structure perception of cell list is announced and its image analysis applications are demonstrated.

Key Words: Sub-Complex, sub-list, connected subsets, neighborhood, continuity, topological mapping, irreflexive, euclidean plane.
1. Introduction

The multiple shades of grey tones in an image is caused by the adjacency relativeness between the grey values that can be articulated as 2-D structures. The structure of a rectangular raster is the simplest and eminent. For human observation, a raster image is appropriate. But image analysis concepts like link of regions or borders or neighbor are characterized indirectly in an image generated by raster. These ideas are the core focus of topology, but is difficult to formulate these designs for a computer.

The modern-day general-topology deliberates the sets of points has the attribute that for any point, for randomly small neighborhood (nbd) of the point there is uncountably many other analogous points. Continuum is an analogous set with uncountable number of elements (emt). On the contrary, any digital image has countable elements. Therefore, a reliable topology of countable sets is desired. Required knowledgebase of analogous topology is scattered making it hard to collect and comprehend. So, implementation of the same into processing digital images or graphics is not complete, forcing the image processing specialists to seek their own solution.

In early 1970, Rosenfeld [2] proposed a digital image can be called as nbd graph if the image pixels are equal to the graph nodes, and the adjacent pixels’ connection to each other equals edges. For the topology of any 2D images, this concept falls for the inconsistencies in the linkage [3, 4]. For e.g., the following comprises of the linkage inconsistency.

![Fig. 1: Validity of the Jordan Theorem in Dissimilar nbd Graphs](image)

According to the Jordan Curve Theorem, in Euclidean plane, a simply-closed curve has exactly two linked components that are inside the curve and the rest of the plane outside the curve. Thus, the rest of the plane outside the curve turns connected when a point is eliminated from the curve.

A similar status quo is expected when the plane is replaced by the raster with a nbd graph on it. An order of adjacent nodes of graph, where every node having just two adjacent ones (Fig-1), states a simple closed path for the finite equivalence of a curve. The rest of the raster is not necessarily always separated into two connected components. Image processing uses 4- nbd and 8-nbd graphs.
as shown (Figs-la & 1c). For 8-nbd, the inner is not separated for a simple closed path from its outer (Fig-lc). For 4-nbd, the inner and outer are detached even after eliminating a path corner (Fig-la). Whereas for 6-nbd, both the properties of a closed path exist (Fig-1b).

Particularly the inconsistencies in the border definitions are disagreeable. In general topology, the region G border comparative to space R is \( \{ c' \} \ni G \) and \( R \setminus G \) are intersected by every nbd \( c' \). Adopting this definition as it is onto a nbd graph, four different definitions are presumed – (i) the inside of the border, (ii) the outside of the border, (iii) the 4-border, and (iv) the 8-border [3].

Figs-2a & 2b shows few of such borders – the border defined for a 4-nbd is not connected and is an 8-connected; the border defined for a 8-nbd is not connected and is a 4-connected. This leads to inconsistent description of head-to-head regions [3]. Furthermore, a border demarcated likewise is a pixel set and so takes a fixed area, then it is a 1D set with 0 area. In general topology, it is never so that the set border and its complement are unalike.

Another struggle is in the disability of incorporating the key topological concepts such as open, continuity, sub-set, mapping, nbd, etc., onto a 2D construction defined by a nbd graph.

According to Rosenfeld [5], usage of dissimilar nbds for back-ground and objects in conjunction with 6-nbd overcomes the disabilities. The idea of the "extended boundary" (Fig-2c) introduced by Pavlidis and Feng [3,6] solve the region adjacency problem; Serra [7] proposed a blend of 4-nbd / 8-nbd; Srinivasan and Elliott [8] wished-for the line sections untying 2 head-to-head pixels from each other termed “border-elements”; Voss and Klette [9] announced the cycles in the nbd graph that uniquely defines its implanting into the plane called "meshes".

However, all these efforts turned out to be flawed solutions and did not lead to a consistent topological theory creating an impression that a "customized" or "alternate" topology does not exist at all. Thus, classical topology is necessary to nail-down the solution.
In the above-mentioned efforts, a digital plane is considered as a building of assorted elements of 0D points, 1D lines, and 2D areas [10,11] is noticed. In topology, it is termed as a Cellular Complex (CC) [1]. A consistent theory using CCs [12] is achieved and is being used in the binary image dispensation and associated arenas.

2. The CC Topology

This considers the assemblies of elements of varied dimensions termed cells [1] that can be presumed as abstract elements’ set or polyhedron sides.

**Definition-2.1**

An Abstract CC $ACC = (S, R, \text{dim})$ is a set $S$ of abstract elements with an anti-symmetric, ir-reflexive, and transitive relation $R \subseteq S \times S$ is termed the Bounding-Face-Relation (BFR) with a dimension function $\text{dim}: S \rightarrow N$ from $S$ into $N$, where $N$ is the set of the integers that are non-negative $\exists \ dim(c') < \ dim(c'') \ \forall$ pairs $(c', c'') \in R$. The BFR $B$ is a poset in $S$ indicating the well-ordered duos of elements $(c', c'') \in S$ is supposed to bound-face $c''$, and is symbolized as $c' < c''$.

![Fig. 3: Samples of (a) 2D Complex, and (b) 3D Complex](image)

A cell binds another cell of bigger dimension $\iff$ it is self-faced in all means. For e.g., in a polyhedron, a point is a 0D side or a line is a 1D side or a square is a 2D side of polyhedron.

Chosen the BFR $R$ to be reflexive, any element must be viewed as a non-proper self-face and $(c', c'') \in R \Rightarrow \ dim(c') \leq \ dim(c'')$.

Nevertheless, it is regarded $R$ as irreflexive.

The dimension of $c'$ is $\text{dim}(c') = d$ and is termed as a $d$-cell. A Complex (Cplx) having dimension $\leq k \ \forall$ elements in it is termed as a $k$-cplx (Fig-3).

In image processing, the 2D area elements are associated with the idea of pixels where the value of a pixel equals the amount of energy emitted from it.
In the hardware, it is observed that the 0D and 1D elements are not signified for image memories and presentations. Hence, the CC implementation was disallowed for image processing. However, lower dimensional elements can be encoded in the same way as the pixels.

**Definition-2.2**

A Sub-Complex (SCplx) \( C' = (S', R', \text{dim}') \) of any Cplx \( C = (S, R, \text{dim}) \) is a Cplx if \( S' \subset S \) and the relation \( R' \) is \( R \cap S' \times S' \), i.e., \( R' \) persists same as \( C' \) ∀ element duos \( \in S' \) and \( \text{dim}'(c) = \text{dim}(c) \cap S' \). Therefore, defining a SCplx satisfies defining the respective subset. Thus, SCplx \( S' \subset S \Rightarrow \text{the SCplx } C' \). All SCplxs of \( C \) may be considered as subsets of \( S \) and thus the concepts of the set theory can be used for operations.

The CC defines consistent topology of finite sets and hence is primarily used in image processing and computer graphics.

A Topological Space (TS) is a duo \( (S, T_{\_}) \) comprising an abstract elements set \( S \) and a system \( T_{\_Y} = \{ T_{\_1}, T_{\_2}, ..., T_{\_i}, ... \} \) of subsets \( T_{\_i} \) of \( S \) called the sub-sets of the space that are open and should gratify the axioms:

**AXIOM-1**

Any \( \emptyset \) subset and the set \( E \epsilon T_{\_Y} \).

**AXIOM-2**

For any \( F \), family of subsets \( T_{\_i} \epsilon T_{\_Y} = U \{ \text{all subsets } \epsilon F \} \) also \( \epsilon T_{\_Y} \).

**AXIOM-3**

If subsets \( T_{\_1}, T_{\_2} \epsilon T_{\_Y} \) then \( T_{\_1} \cap T_{\_2} \epsilon T_{\_Y} \).

A TS will have the separation property \( P_0 \) if it satisfies the additional axiom stated below:

**AXIOM-4**

For any two elements \( e_{\_1}, e_{\_2} \epsilon E, \exists \) in \( T_{\_Y} \) \( \exists \) an open subset \( T' \) with only one element \( \epsilon T' \).

In Axiom-4, \( \exists \) a subdivision of the whole space into two parts \( T' \) and \( S \backslash T' \) \( \exists \) one element \( \epsilon T' \), the other in \( S \backslash T' \), for any two \( c_{\_1}, c_{\_2} \epsilon S \), and one part is open.

A TS is **finite** if the set \( S \) \( \subset \) countable elements. Notice that Axiom-2 is true for both countable and uncountable; Axiom-3 holds good only for \( \cap \) (countable subsets \( T_{\_i} \)), weakest form of Axiom-4 is very important, and is termed as \( T_0 \)-axiom. Therefore, a space sufficing Axiom-4 possess the \( P_0 \) and is commonly termed as \( T_2 \)-spaces or Hausdorff spaces in general topology.

**Definition-2.3**

Sets \( S_{\_1} \) and \( S_{\_2} \) have the same cardinality (size) if there is a one-to-one correspondence function \( f: S_{\_1} \rightarrow S_{\_2} \), denoted as \( \text{CARD} (S_{\_1}) = \text{CARD} (S_{\_2}) \).
THEOREM-1

According to Definition 2.1, Each finite TS with the $P_0$ is isomorphic to ACC.

Proof

An open subset comprising any emt $c'$ is called the neighborhood of $c'$, denoted by $\text{nbd}(c')$, which is open. According to Axiom-3, $\cap \text{nbd}(c')$ in a finite space is an open subset and is called the smallest nb of $c'$ and denoted by $S_N(c')$.

Consider the smallest nbds $\forall$ emts $\in S$. Firstly, suppose that $\exists$ an emt $c_1$ with smallest nbd $S_N(c_1) \subset \exists$ one emt $c_2 \neq c_1$:

$$S_2 \in S_N(c_1) \quad \text{and} \quad c_2 \neq c_1$$

------------------------(i)

$$S_N(c_1) = \cap \text{nbd}(c_1) \quad \text{and} \quad c_2 \in \cap \text{nbd}(c_1) \quad \text{From Axiom-4, } \exists \text{ an nbd}(c_2) \not\supseteq c_1. \quad \text{Thus}$$

$$c_1 \not\in S_N(c_2)$$

------------------------(ii)

(i) & (ii) signifies a BFR $B$ among $c_1$ and $c_2$ which is an anti-symmetric and ir-reflexive. To show transitivity, consider $c_1$, $c_2$, and $c_3 \not\supseteq B$ is true for $(c_1, c_2) \& (c_2, c_3)$.

$$c_2 \in S_N(c_1) \quad \text{and} \quad c_1 \not\in S_N(c_2)$$

------------------------(iii)

$$c_3 \in S_N(c_2) \quad \text{and} \quad c_2 \not\in S_N(c_3)$$

------------------------(iv)

From Axiom-3, the intersection $S = S_N(c_1) \cap S_N(c_2)$ is an open subset and $\subset c_2$. So, it is an open nbd$(c_2)$ and $\not\subset S_N(c_2), \text{ since } S_N(c_2) \subset \text{nbd}(c_2)$. Henceforth

$$S = S_N(c_2) \quad \text{and} \quad S_N(c_2) \subset S_N(c_1)$$

------------------------(v)

(iv) and (v) signifies $c_3 \in S_N(c_1) \text{ and, from Axiom-4, } \exists \text{ an open subset } S' \supseteq c_3 \text{ in } S' \text{ but } c_1 \not\in S'$. The set $S_N(c_3)$ is by definition the intersection of all open subsets containing $c_2$. So

$$S_N(c_3) \subset S' \quad \text{and} \quad c_1 \not\in S' \Rightarrow c_1 \not\in S_N(c_3)$$

Hence, $B$ is true for $(c_1, c_2)$ proving the transitivity of $B$. From Definition 2.1, any anti-symmetric, ir-reflexive, and transitive relation is a BFR. Then $\dim(\cdot)$ is defined in any way providing:

$$\forall c_1, c_2 \in E \quad \text{and} \quad (c_1, c_2) \in B \Rightarrow \dim(c_1) < \dim(c_2)$$

Means of finding a real dim $(\cdot)$ [13] entails in the cardinality of $S_N$. From Definition 2.3,

$$\dim(c') = \max(c) (\text{CARD}(S_N(c))) \cdot \text{CARD}(S_N(c'))$$

------------------------(vi)

Thus $c_1 < c_2 \Rightarrow S_N(c_2) \subset S_N(c_1)$ and $\text{CARD}(S_N(c_2)) < \text{CARD}(S_N(c_1))$.

Thus, (vi) satisfy Definition 2.1. Therefore, a countable TS having $\geq$ one $S_N$ with $>1$ emt is isomorphic to ACC. Complementarily, $S_N(c_1) = c_1$ is true $\forall c_1 \in E$. Thus, all emts are open subsets and that TS is named as Discrete Space.
\( \exists c_2 \) in satisfying (1) and (2) with \( c_1 \). Then, \( B = \emptyset \) corresponds to a CC with no mutually bounding emts. For example, it is a 0D Cplx \( \supseteq 0D \) emts alone. Consequently, a countable TS is isomorphic to \( ACC \).

From Definition 2.1, every CC is a topological space. From the theorem, \( S_{N}(c_1) \subseteq c_1 \) itself and all emts bounded by \( c_1 \) gives the definition of subsets that are open in a CC.

**Definition-2.4**

A SCplx \( S \) of \( C \) is named open in \( C \) if \( \forall c' \in S \), all emts \( \epsilon \) \( C \) are face-bounded by \( c' \subseteq S \).

E.g., In the complex of Fig.3a, \( S_1 = \{ x_1, h_3, x_2 \} \) is open but \( S_2 = \{ h_4, j_3, h_5 \} \) is not open because \( S_2 \notin x_1 \) face-bounded by \( h_4 \).

The subset \( S_i(c', C) \subseteq c' \) is open and all emts \( \epsilon C \) face-bounded by \( c' \) is named as Open-Star (\( O^* \)) of \( c' \in C \) which is \( S_N(c') \) of \( c' \in C \). The \( S_i \) is the meekest subset open in a Cplx, and rest of all subsets that are open = \( \cup O^* \). Fig.4 shows the \( S_i \) in different dimensional Cplxs.

Hence, note that the open subsets defined thus satisfies the axioms of the topology. Therefore, any Cplx is a TS. All perceptions and outcomes are expressive for Cplxs and is reliable. It can be demonstrated that the behaviors of all the ironies and inconsistencies of the nbd graphs disappear.

### 3. Solution of the Ironies

Trivially, the 4-nbd and the 8-nbd lead to the linkage irony [3, 4], but the 6-nbd is not. Hence for any reliable countable TS all 2C has six 2Cs as neighbors.

All Cplxs with 2Cs can have random count of neighbors lacking any inconsistencies.

Hence, the basis of the irony is that the nbd graphs are not covenant with the topological axioms, and not because of the number of neighbors. The topological Cplx incapacitates the irony [11, 12] in the following means. Consider the idea of topological linkage used on CCs as shown below.

**Definition-3.1**

An arrangement of elements starting at \( c' \) and concluding with \( c'' \) of a subset \( S \) of a Cplx \( C \) is termed as path from \( c' \) to \( c'' \) in \( S \), if \( \forall \) two emts that are head-to-head in the arrangement and face-bounding one-another.

**Definition-3.2**

Any subset \( S \) is termed as connected, if \( \exists \) a path \( c' \rightarrow c'' \) for any \( c', c'' \in S \).

In Fig.5, apply Definition 3.2 that was constantly cast-off to reveal the linkage
irony [3, 4].

Fig. 4: $O^* \text{Sr}(c^k, C^d)$ of several $c^k \in C^d$ of dim $d = 1, 2, & 3$.

In Fig-5, consider a Cplx $O \subset 2$-element $A, B, C, D$. From Definition 3.1, a SCplx $S \subset A$ and $D$, but $S \not\subset B$ and $C$ is linked $\Leftrightarrow$ the 0-emt $p \in S$ since a path in $S$ from $A \to D$ have to pass through $p$. If $p \in S$, then $p \notin OS \Rightarrow OS$ is detached.

Thus, for $S = 0$-emt $p$ or $OS = 0$-emt $p$, no irony arises. However, the association of $p \in S$ remains undefined then both $S$ and $OS$ are linked or detached. Hence, the irony is caused by ignoring the 0-emt.

To explain what is causing the irony when using a nbd graph, the below Definitions 3.3 and 3.4 are needed.

**Definition-3.3**

Two nD CCs $C = (E, P, \text{dim})$ and $D = (E', P', \text{dim'})$ are called dual, $\langle C|D\rangle$, if $\exists$ 1-1 mapping from $E \to E'$ with reverse BFR. If $c_1 \leftrightarrow p_1$ and $c_2 \leftrightarrow d_2$ then $c_1 < c_2 \Rightarrow p_2 < p_1$ and dim$(p_i) = n - \text{dim}(c_i), i = 1,2$.

**Definition-3.4**

A kD SCplx $S$ of an nD CC $C = (E, P, \text{dim})$ with $k < n \subset E$ with $\text{dim} \leq k$ is
named a $kD$ scaffold of $C$. Also, $\forall$ $n$-Cplx, $\exists$ only one dual $n$-cplx and only one $kD$ scaffold $\forall k < n$.

A nbd graph is a 1D Cplx, the NG4, a 4-nbd graph, is a 1D scaffold of the 2D Cplx dual to CR (Fig-6a).

From Fig-6b, a SG of NG4 is measured by $V$, a subset of the vertices, and the edges whose end-vertices $\in V$. This kind of SG specifies a $<CR\mid S> \ni <V\mid \text{pixels}>$ and $<\text{edges(SG)}\mid l\text{-emts}>$. Two pixels of $S$ are untied by $l$-emts. The NG4 and SG is 1-scaffold $\ni <0\text{-emts(CR)}\mid 2\text{-emts}>$. Hence the affiliation of $0\text{-emts} \in S$ remains undefined and $\notin S$ or $\notin$ its complement. Hence, $0\text{-emts}$ diagonal links cannot be denoted.

![Fig. 6: Correlation between 2-cplx Denoting CR and NG4](image)

This faintness is satisfied for the 8-nbd graph by inserting in the 1-scaffold, the diagonal edges that match to no $l$-emts of the given Cplx CR. Their role is to signify the nbd of the 2-emts with a common $0$-emt is not satisfactory. Fig-7 signifies the illusory TS. Fig-5 shows method to deform the Cplx in harmony with the NG8. The 2-emts A and D has a common $1$-emt $L_1$, and the 2-emts B and C has another common $1$-emt $L_2$ without intersecting $L_1$. $L_1$ and $L_2$ are insincerely implanted into the construction to recompense the non-appearance of the common face $0$-emt $p \forall A, B, C, D$.

![Fig. 7: The Unreal Topological Construction Corresponding to that of the NG8](image)

The Fig-7 construction is topologically incredible. The reason for the NG6 free of ironies is when a 2-cplx has the distinctiveness for any two 2-emts having a common face that is also a $1$-face, $\forall$ $0$-emt bounds $\leq$ three $l$-emts. So, there is no need to specify the affiliation of the $0$-emts. The affiliation of the $l$-emts
defines the linkage as totally stated by a NG. Thus, a NG denotes the linkage of a 2D construction ⇔ the matching SCplx is strongly linked.

Obviously, the linkage irony vanishes if every subset \( S \) of the Cplx CR denoting the raster is properly defined. Hence, the right way of defining is through affiliation predicate \( M(S,c) \) where any nD space \( c \in S \) or \( c \notin S \).

Particularly, when \( N \) subsets, \( \{S_i\}, i = 1, 2, 3, \ldots, N \), comprise a separator of CR is reasonable. Thus, it is understood that any affiliation predicate as coloring of every nD space element by one of the \( N \) colors.

![Fig. 8: Sample of a Reliable Scripting of Linked Black and White Diagonals](image)

In Fig-8, none of the nbd relations is able to define the borders of the V-shaped regions, \( \exists \) the border of V is linked and similar sequences of outlining procedure is possible for both cases.

It can be achieved defining the cells shown by white circles (WC) as affiliates of white (non-striped) V region, and black circles (BC) as affiliates of the black (striped) V region. So, each border is outlined \( \exists \) cells with the selected region color always persist to the RHS of the outlining. As only the WC are linked to one-another and the BC are detached, no linkage irony for white V. And similarly, for black V.

Thus, the irony vanishes if a nbd relation to pairs of same dimensional space elements is force ceased. The most reliable means of postulating the linkage of subsets of any image is to represent the image as a CC and define the affiliation of all nD cells here. Such a depiction of images does not only eliminate the inconsistencies, it also creates the possibility to effectively denote the construction of images with minute details as in Fig-8. The procedures for an applied comprehension of this method are discussed in the following section.

Now Definition-3.5 is required to overcome the inconsistencies in defining the boundaries.
Definition-3.5

The sub-set \( S \subseteq C \) comparative to \( C \) has border as the sub-set \( F_j(S,C) \) \( \forall c' \in C \ni \) any open \( \text{nbd}(c') \supseteq \) elements of both \( S \) and \( C \).

From Definition 3.5, an open \( \text{nbd} \) can be easily substituted by any \( O^* \), \( S(c', C) \). Thus, from Fig-9a, the subset \( S \supseteq \{ l_1, l_2, a_1, a_2, a_3, p \} \), reveals the border in the Fig-9b by thick lines and also \( O^* \) of borderline emts satisfying the Definition 3.4.

A unique definition of the border exists under the CCs topology. Distinguishing between the inner and outer borders are not required and so is with the NG. The border has no end points similar to the border in Fig.9b.

![Fig. 9: Sample of (a) SCplx and its (b) Border](image)

Hence the claim of assuming a 0-cell as the border end point of a subset \( S \) and seeing the possible affiliations in \( S \) of the 4 pixels circumscribed by this. From Pavlidis [3], the idea of head-to-head regions can be replaced by incident regions.

**Definition-3.6**

The SCplxs \( S_1 \) and \( S_2 \) of a Cplx \( C \) are called *incident* mutually if there is no intersection and \( c_1 \in S_1 \) and \( c_2 \in S_2 \) they bound one-another.

Now deliberate the border area issue. The border of a 2D subset \( S'' \) of a 2D Cplx \( C'' \) contains no 2-emts, as the \( O^* \) of \( c'' \supseteq \{c''\} \) and hence does not intersect \( S'' \) and \( C'' \setminus S'' \).

So, this kind of border \( \supseteq \) emts like \( 1D \) line and \( 0D \) points \( \Rightarrow \) \( 1D \) SCplx of \( C'' \). Hence, a non-zero area is assigned to \( 2D \) pixel. Then, area of a \( 1D \) Cplx = 0. Finally, from Definition 3.6, the border is the same for a subset \( S \) and \( C \setminus S \), since the symmetricity w.r.t \( S \) and \( C \setminus S \).

The CCs being consistent it eliminates all topological ironies and inconsistencies from the philosophy of digital images. This concept has numerous advantages and can be used to describe the construction of nD images.
Definition-3.7

A 2D raster image with algorithm pertinent to multiple values and nD images in a raster topology, the nD image is considered as an array, Image[N]. Thus, for 2D image, the pixel at \((r,s) = \text{Image}[s.NP+r]\) where NP = \#pixel of a row\.

Given a binary array Image\([N]\), the 1\(^{\text{st}}\) one returns the \#neighboring pixels\ contingent to the pixel color; the 2\(^{\text{nd}}\) is the index of \(k\)^{th} neighbor of \(j\)^{th} pixel. Thus, each pixel-tag = own-tag + tag(linked emts \(\epsilon\) pixel). Hence, Memory provision for the array Tag\([N]\) \(\geq 1/8 \times (\log_2N)\) bytes, where \(N = \#\text{image emts}\) and size of Tag\([N]\) = Image\([N]\).

4. Encrypting Images on CCs

A raster image \(M\) is well-defined by \(M: e \rightarrow l\), where \(e = \text{raster elements}, l = \text{tags}\). The tags are grey-value symbols of subsets. Thus, the raster is represented as a 2D \(CC \supseteq\) three different dimensional elements. The 2\(-\text{emt}\) pixels, the 1\(-\text{emt}\) and 0\(-\text{emt}\) are faces of the pixels.

An image encrypting \(\Rightarrow\) encrypting all the affiliated subset elements. The pixel encrypting must be comprehended by a memory space allotted to every pixel and the tag of the subset \(\Rightarrow\) the pixel. The lower-dimensional elements encrypting is done the same way. Mostly, a fraction of the memory term is sufficient to encrypt the element affiliation. For \(N_e\) sub-sets to be differentiated in the image, it requires \(K \geq 1/4 \times \log_2N_e\) bytes/element.

For any image, only 2 sub-sets are differentiated and hence only \(1/4\) byte/element is required. From Euler theorem, the number of the 0\(-\text{emts}\) in the 2D raster without borders = the count of the pixels, \(N_p\). So, \(#\text{1-emts}\) = \(2 N_p\).

By Definition 3.7, for 2\(-\text{emts}\), the subset affiliation of all elements of a 2D raster when encrypted requires an overall memory = \(4N_p \times \log_2N_e = 4\) times more than pixel encrypting. For an efficient encrypting of the set affiliation, introduce a Global Face Affiliation (GFA), that is a regulation postulating the set affiliation of every 2\(-\text{cell faces}\) as \(f(2\text{-cell})\) that cannot be randomly chosen, but represented as an affiliation predicate. For example, specifying all 2\(-\text{cell faces} \supseteq S\) as the 2\(-\text{cell}\) is not legalized because it may be equal to \(f(\text{two 2-cells})\), where one \(\epsilon S\), and other \(\epsilon C\backslash S\).
Fig. 10: One Possible Allotment of 0-D & 1-D Elements to the 2-D.

Fig-10 shows that a dependable GFA is built by equating every 0-\textit{emt} and 1-\textit{emt} to a uniquely defined 2-\textit{emt}. The regulation declares all 0-\textit{cell} and 1-\textit{cell} ε {exclusively well-defined 2-cell}. Hence such a GFA ⇔ declaring a NG6 of the 2-cells.

The other two nbds are typical cases of GFA. The NG4 postulates that the 0-\textit{cells} ∉ any subset, and is not possible. The NG8 resembles double 0-\textit{cells} and exists as duplicates, one ε the same subset as the pixel lying north-west of it and the other to north-east.

This leads to inconsistencies as the two pixels ε dissimilar sub-sets. Consequently, these global regulations and the corresponding nbds conflict, evade using it. A practically useful regulation is used considering the pixel tags as elements of an ordered set.

**The Maximum Tag Regulation (MxTR)**

All elements \(c'\) with 0D or 1D has \(Tag = \text{Max}(\text{Tags(every pixels face-bounded by } c'))\).

Fig.5 establishes this regulation when 2 pixels in a diagonal duo (A & D) in \(O^\star(P)\) has the same tag \(L\) with grey-value ≠ Tags(other 2 pixels), then the pair is linked ⇔ \(L > \text{Tags(diagonal duo (B & C))}\). In this case, the diagonal duo (B & C) are detached. From Rosenfeld [5], using the NG4 for the background and the NG8 for the objects is a specific case of the MxTR and the advantage the validity to images that are multiple-valued and not binary.

The main disadvantage of this regulation - the 1-pixel-width slanted-narrow stripes are detached when grey-levels < background grey-levels. Introducing a Minimum Tag Regulation (MnTR) also experiences the same problem. For a NG6, the above-mentioned global rule has a similar drawback characteristic. From Fig-10, left-slanted-narrow-stripes are linked but right-slanted are detached. It is possible to make the right-slanted-stripes as linked and left-
slanted as detached. In the same image, none of the above said regulations represent all narrow-stripes as linked (be it dark / light, left / right slanted). Fig-8 shows such an image. So, in the image, a more Cplx GFA regulation may be applied provisioning an approximate solution to the problem of reducing the linked regions count.

The Equnal-Regulation

All 1-emt $Q'$ assumes Tag = $\text{Max}(\text{Tags}(2 \text{ pixels face-bounded by } Q'))$. The tag of 0-emt $q'$ is named as $O^*(q') \supset$ just one pixel-duo of a diagonal $\Leftrightarrow$ tags. For two such duos with only one $\epsilon$ narrow-stripe then the Tag(other pair) $\rightarrow q'$. Otherwise the MxTR or MnTR is applied and the lighter duo is linked. From testing a nbd($q'$) $\supset$ 2x2 pixels shows it does not suffice in deciding if a pixel-duo $\epsilon$ narrow-stripe or not. The nbd $\supseteq$ 4x4 = 16 pixels. Fig-8 shows the result of this regulation.

![Image](image.png)

Fig. 11: Sample Showing the Need for Explicit Face Affiliation Encrypting

GFA regulation allows saving memory space encoding only the pixels. The affiliation of the faces of pixels are stated indirectly and memory space is not needed. Though, the existing global regulations signify only limited affiliation conditions of the faces of pixels with conditions where global regulations do not apply and explicit encrypting of the face affiliation is required.

Consider high resolution binary image with little details in Fig-11a which can be encrypted using some global regulation. Hence, explicit encrypting of the pixels alone is needed and 1/8 byte/pixel is enough to suitably specify the linked region. It has twelve rows and twenty-four columns $\Rightarrow$ total memory = 1/8 x 12 x 24 = 1/8 x 288 = 36 bytes.

Fig-11b signifies bi-directional 2-factor compressed image of Fig-11a. The image resolution crooked smaller and not all global regulation may suitably characterize all the B&W regions links. The image can be suitably encrypted by explicit 0-emts or the EQUINAL regulation. The 1-emts needs to be defined as
their affiliation as the incident pixels and require no explicit encrypting. So, based on the Fig-10 assignment regulation, explicit encrypting of 0-emts requires 1/4-bytes/pixel = ¼ for the pixel + ¼ for the 0-ent assignment. So, as in Fig.11a, the requirement of total memory = 1/8 x (2 x 12 x 6) = 1/8 x 144 = 18 < 36 bytes.

After compressing the image again, correct encrypting of the region linkage is possible when 0-emts and 1-emts are explicitly encrypted (Fig-11c). Now, as in Fig-11b, 1/2-bytes/pixel is required, so, the total memory = 1/8 x (4 x 6 x 3) = 1/8 x 72 = 9 < 18 bytes.

This shows that the need for encryption is justified, and saving 75% memory space is advantageous. For high resolution, a GFA may be used without any extra memory space. For low resolution, extra memory space is needed enabling to correctly represent fine details.

5. The Data Structure–Cell-List

The study has considered any image as a 2D CC with tags allotted to all emts originating from the mutable grey-values measured while scanning an image, i.e., an image rarely gets linked region as 2D SCplx which is open and linked, having constant value pixels.

Alternatively, image regions can be 2D blocks, having a semantic meaning. Segmentation, which is out of scope of this article, with objective of considering encryption of segmentation results, finds such regions and tags all elements. The image segmentation w.r.t CCs is splitting an image into cell-blocks with certain features predefined by "uniformity predicate" [3].

The border of any region is a 1D SCplx having branching-points (0D blocks) with 3 or more regions meeting. Portions of border parted by branching-points are the 1D blocks also known as lines or border-segments. If the border does not have any branching-points, all linked border-segments is a line.

There exists a BFR among the blocks with branching-points and lines. Thus, these object collections are a new CC with regions=2-cells, lines=1-cells, branching-points=0-cells. The original Cplx are fragmented to obtain such new cell-blocks. So, as shown in Fig.12, such a new Cplx shall be called the Complex Block (BC) with the Cells Block (BCL) as elements.
Fig. 12: Sample of a BC. Ri = Regions, Li = Lines, Pi = Points

A BC can be considered as a universal idea of head-to-head RG [3,5]. The 1D scaffold of <BC|Cplx >.

A BC may be described by a data structure, cell list, consisting 1 metric and 3 topological sub-lists. The topological sub-lists are the 0D, 1D, and 2D BCLs. The cell list information must be enlarged by the sub-list metric containing few 1-cells and 0-cells co-ordinates, facilitates investigating and reestablishing the original image.

The few 1-cells are identified by softening the Li as Digital-Straight-Line-Segments (DSS) providing an efficient polygonal depiction of the borders. The encrypting of DSS succeeded in exact reconstruction of the original borders. Thus, the coordinates of the DSS end points has some additional parameters specifying which DSS runs through the which end-points. Thus, polygonal depiction is the correct and cost-effective encryption of borders [14]. Fig-12 explains the construction of cell list, and are shown in the following Tables.

<table>
<thead>
<tr>
<th>Branching Points List</th>
<th>No.</th>
<th>X</th>
<th>Y</th>
<th>P1</th>
<th>P2</th>
<th>P3</th>
<th>P4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Co-ordinates</td>
<td></td>
<td></td>
<td></td>
<td>10</td>
<td>30</td>
<td>17</td>
<td>24</td>
</tr>
<tr>
<td>Line</td>
<td></td>
<td></td>
<td></td>
<td>24</td>
<td>23</td>
<td>17</td>
<td>20</td>
</tr>
<tr>
<td>East</td>
<td></td>
<td>-L3</td>
<td>0</td>
<td>-L4</td>
<td>-L5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>West</td>
<td></td>
<td>0</td>
<td>+L5</td>
<td>0</td>
<td>+L4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>North</td>
<td></td>
<td>+L2</td>
<td>-L2</td>
<td>+L6</td>
<td>-L6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>South</td>
<td></td>
<td>-L1</td>
<td>+L1</td>
<td>+L3</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1st row (Table 1) is the 0-cells pointers used in the Fig.12. The next 2 rows are the points’ coordinates. The next 4 rows are the pointers of lines bound by 0-cell and directed with a minus-sign as line-start and a plus-sign as line-end.
Table 2

<table>
<thead>
<tr>
<th>No.</th>
<th>L1</th>
<th>L2</th>
<th>L3</th>
<th>L4</th>
<th>L5</th>
<th>L6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point</td>
<td>Begin</td>
<td>P1</td>
<td>P2</td>
<td>P1</td>
<td>P3</td>
<td>P4</td>
</tr>
<tr>
<td></td>
<td>End</td>
<td>P2</td>
<td>P1</td>
<td>P3</td>
<td>P4</td>
<td>P2</td>
</tr>
<tr>
<td>Region</td>
<td>Right</td>
<td>R1</td>
<td>R1</td>
<td>R2</td>
<td>R2</td>
<td>R2</td>
</tr>
<tr>
<td></td>
<td>Left</td>
<td>R2</td>
<td>R3</td>
<td>R3</td>
<td>R4</td>
<td>R4</td>
</tr>
<tr>
<td>Metric</td>
<td>Begin</td>
<td>1</td>
<td>5</td>
<td>9</td>
<td>12</td>
<td>14</td>
</tr>
<tr>
<td></td>
<td>End</td>
<td>4</td>
<td>8</td>
<td>11</td>
<td>13</td>
<td>15</td>
</tr>
</tbody>
</table>

1st row (Table 2) is the pointers of the line. The next 2 rows are the end points of the line. The pointers are zero at any branching-point for a closed and non-starting/ending line. Next 2 rows are pointers to the regions in RHS / LHS of the line. Last 2 rows are pointers to the end point coordinates of DSS of the line of the subsequence of the metric list. Each coordinate pair has additional parameters for precise rebuilding of DSS that runs through the end-points.

Table 3

<table>
<thead>
<tr>
<th>No.</th>
<th>R1</th>
<th>R2</th>
<th>R3</th>
<th>R4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tag</td>
<td>0</td>
<td>5</td>
<td>9</td>
<td>0</td>
</tr>
<tr>
<td>Border Start Line</td>
<td>+L1</td>
<td>+L3</td>
<td>-L5</td>
<td>-L6</td>
</tr>
</tbody>
</table>

1st row (Table 3) is the pointers, 2nd row is the grey-value-tags, and last row is a line pointer in the region border. It is possible to rebuild the comprehensive arrangement of the lines constituting the border if started from this point in the right direction. The borderline is directed in a way that the region is always on RHS. The minus-sign guides reverse-traversal of starting line from end to start to get the right path of the border.

Table 4

<table>
<thead>
<tr>
<th>Identifier</th>
<th>01</th>
<th>02</th>
<th>03</th>
<th>04</th>
<th>05</th>
<th>06</th>
<th>07</th>
<th>08</th>
</tr>
</thead>
<tbody>
<tr>
<td>Identifier</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>17</td>
</tr>
</tbody>
</table>

6. Applications

Since 1985, this technique has been used for solutions of various snags in analyzing images. The image should be segmented into hundreds of uniform regions before computer processing. Each region pixel has same tag, dissimilar than the neighboring region’s tags. Up to 256 tags can be used. Using a computer program, an image is converted into a cell-list. The computer scans any image line-by-line from top-left corner to the bottom right corner of the image identifying new borders, and traces border recognizing the DSS and writing information into the cell-list. To accelerate the tracing, label all 0-emts of the given image by the Direction Change Tags (DCT). The DCTs guides the next step’s locus, the branching-points and the "north-west corners" as possible location of new border while scanning and tracing borders.
Because of the DCTs, the border tracing cell-list has a high data-compression ratio (~ 20 times). The resultant cell-lists has neighbor regions, mutually-contained regions, incident lines and points, and it makes any analysis of the images efficient and simple. All geometrical calculations and transformations can be performed by re-calculating the coordinates and the intermediary points.

For reverse engineering of the transformed digital image into raster form, the cell-list can be transformed into the original form. To achieve this, the DSS constituting the borders are drawn and the regions are packed with the original tags. Fig.13 shows the sample of an image accurately rebuilt from a cell-list.

If the borders of the original image are worn by clatter they must be sharpened. This can be achieved by the polygonal approximation with a random pre-defined allowance. Using the cell-list permits realizing it faster without calculating border emts but only for metric-list end-points of the DSS.

Similarly, the central axis of a stripe can be determined faster while analyzing the DSS end-points’ coordinates instead of the border elements. This is more precise and flexible. The structural analysis of image and recognition of objects [15] are done through cell-lists. The analysis of the co-ordinates of polygon vertices has high and reliable recognition to the geometrical transformations with slight disruptions of the borders.

This technique of recognition is tagged SG isomorphism. The most important modifications follow:

(i) The construction of an image is through CC and BC instead of graphs.

(ii) The region-tag of the Image Under Analysis (IUA) is denoted by geometrically precise cell-list ⇒ the region and its nbd can be accurately rebuilt.

(iii) The agreement among the IUA and the object-models under recognition is tested with predefined 1-place and 2-place region predicates and rest of the image. The predicates are specified in the class definition as they are different for different objects.

(iv) The class definitions under recognition are hierarchically organized.

Fig. 13: Sample of An Image Rebuilt from Cell-list
(v) The class definitions are interactively determined as knowledge base tables stipulating the necessary 1-place and 2-place predicates to compose an object of the class definition.

This technique can be used for the recognizing the handwriting and sketches [15].

7. Conclusion

The engineering of the prevalent topological concepts of the cplxs brings several advantages for the following:
(a) Precise and cost-effective description of the images

(b) Quick and topologically-credible image transformations, and

(c) Implementation of elastic and persistent methods for organized analysis of images.

8. Abbreviations

- Complex Block – BC
- Cell Block – BCL
- Cellular Complex – CC
- Complex – Cplx
- Image Under Analysis – IUA
- Global Face Affiliation – GFA
- n-Cell – nC
- Element – emt
- Neighborhood – nbd
- Sub-Graph – SG
- Direction Change Labels – DCL
- Minimum Tag Regulation (MnTR)
- Maximum Tag Regulation (MxTR)
- X-Nbd Graph – NGX

References


