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Existence of Solution of Global Cauchy Problem for Some Fractional Abstract Differential Equation

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Abstract

In this paper, by using the fixed point theory, applications of the global condition in physics and new methods, investigation is made and prove the existence and uniqueness of solutions of global Cauchy problems for the fractional differential equation $D^{\alpha} y(t) = g(t, y(t)), t \in L := [0, t_1],\ y(0) + f(y) = y_0$ in a Banach space and obtain two new results.

Key Words: Fractional abstract differential equation, global cauchy problems, schauder fixed point theorem.

1. Introduction

In this paper, the existence of solution for the following global Cauchy problem in Y and its uniqueness using contraction mapping principle is considered and proved.

$$\begin{cases} D^a y(t) = g(t, y(t)), t \in L := [0, t_1], \\ y(0) + f(y) = y_0 \end{cases} \quad (1.1)$$

where $(Y, \|.\|)$ is a Banach Space, $0 < a < 1, D^a$ are the a^{th} Caputo fractional derivatives and $g, f : L \times Y \to Y$ are appropriate functions to be specified later.

There are many results for existence and uniqueness results for the global Cauchy problem.

The following conditions ([1]-[5]) have been satisfied.

(A1) For all $t \in L$, the function $g(t,.) : Y \to Y$ is continuous and for each $y \in C := C(L, Y)$, the function $g(., y) : L \to Y$ is strongly measurable and there exist $c, n > 0$ ∍ for each $y \in Y$, $\|g(t, y)\| \le m\|y\| + h$ for all $t \in L$.

(A2) There exist $n > 0$ such that for any $y,\ x \in Y, \|g(t, y) - g(t, x)\| \le n\|y - x\|$ for all $t \in L$.

(B1) $f \in C(C, Y)$ and there exist constants $N, H > 0$ such that for each $y \in C$
$\|f(y)\| \le L\|y\|_0 + H$
where $\|y\|_0 = \sup_{t \in L} \|y(t)\|$.

(B2) There exist a constant $N > 0$ such that for any $y, x \in C$,
$\|f(y) - f(x)\| \le N\|y - x\|_0$.

For the existence and uniqueness, the following condition should be satisfied

$$(\mathrm{I}') \quad N + \frac{n t_1^a}{\Gamma(1+a)} < 1.$$

In this paper, the condition (I') has to be improved and it is worth putting of the improvement instead of present.

The existence of unique solutions for the problem can be proved using Schauder fixed point theorem and new methods

Let Y be the Banach space consisting of all functions $y \in C'$ with the norm $\|y\|_0 = \sup_{t \in L} \|y(t)\| < \infty$

$$y(t) = y_0 - f(y) + \frac{1}{\Gamma(a)} \int_0^t (t-u)^{a-1} g(u, y(u)) du,\ t \in L. \quad (2.1)$$

Using the above results and the new assumption, the following theorems can be proved as follows:

Theorem 1.1

Under (A1), (B1) and the condition (I) $N < 1$

Problem (1.1) has at least one solution.

Proof:

Let M, $a_1 > 0$ be two constants with

$$e^{\frac{t_1^{a_1}}{M^{\frac{2a_1}{a}}}} \leq \frac{4}{3+N}, \quad a > a_1 > 0,$$

$$\frac{6nt_1^a}{a_1 M^{1+a_1}\Gamma(a)} + \frac{2nt_1^a}{a_1^2 M^{2\left(\frac{1-a_1}{a}\right)}\Gamma(a)} \leq \frac{1-N}{4} \tag{2.2}$$

$$\left[1 + \frac{2t_1^a}{M^{\frac{2a_1}{a}}}\right]\left[N + \frac{2nt_1^a}{M^{2+a}\Gamma(1+a)}\right] < \frac{1+5N}{6} \tag{2.3}$$

Let D be a closed, bounded and convex subset of Y and the operator Δ as follows ([6],[7]):

$$D = \left\{ y \in Y : \|y(t)\| \leq K e^{\left(t - \frac{(j-1)t_1}{M^{\frac{2}{a}}}\right)^{a_1}}, t \in \left[\frac{(j-1)t_1}{M^{\frac{2}{a}}}, \frac{jt_1}{M^{\frac{2}{a}}}\right]\right.$$

$$\left. j \in \left\{1, 2, \ldots\ldots\ldots\ldots, M^{\frac{2}{a}}\right\} \text{ (or) } t \in \left[\frac{\left(M^{\frac{2}{a}}\right)t_1}{M^{\frac{2}{a}}}, t_1\right]\right\} \tag{2.4}$$

where $\left[M^{\frac{2}{a}}\right] = \max\left\{m : m \leq M^{\frac{2}{a}} \text{ and } m \text{ is an integer}\right\}$.

$$K = \frac{3}{1-N}\left(\|y\|_0 + H + \frac{ht_1^a}{\Gamma(1+a)}\right) \tag{2.5}$$

and

$$\Delta y(t) = y_0 - f(y) + \frac{1}{\Gamma(a)}\int_0^t (t-u)^{a-1} g(u, y(u))du, t \in L \tag{2.6}$$

Hence it is proved through the following three steps.

STEP: 1

From (2.2) to (2.6), (A1), (B1) and for each $y \in D$, the result is

$$\|\Delta y(t)\| \leq \left\| y_0 - f(y) + \frac{1}{\Gamma(a)}\int_0^t (t-u)^{a-1} g(u, y(u))du \right\|$$

$$\|\Delta y(t)\| \le \|y_0\| + NK\, e^{\left(\frac{t_1}{M^{\frac{2}{a}}}\right)^{a_1}} + H + \int_0^t \frac{(t-u)^{a-1}}{\Gamma(a)}(n\|y(u)\|+h)du.$$

$$\le \|y_0\| + NK\left[1+\frac{2t_1^{a_1}}{M^{\frac{2a_1}{a}}}\right] + H + \frac{ht_1^{a}}{\Gamma(a+1)} + \frac{nK}{\Gamma(a)}\int_0^t (t-u)^{a-1} e^{u^{a_1}} du$$

$$\le \frac{1-N}{3}K + NK\left[1+\frac{2t_1^{a_1}}{M^{\frac{2a_1}{a}}}\right] + \frac{nK}{\Gamma(a)}\int_{\frac{(M-1)t}{M}}^{t} (t-u)^{a-1} e^{\left(\frac{t_1}{M^{\frac{2}{a}}}\right)^{a_1}} du$$

$$+\frac{1}{\Gamma(a)}\int_0^{\frac{(M-1)t}{M}} (t-u)^{a_1-1}(t-u)^{a-a_1} e^{u^{a_1}} du$$

$$\le K\left[\frac{1+N}{2} + \frac{2nt_1^{a}}{a_1 M^{1+a_1}\Gamma(a)}\right]$$

$$\|\Delta y(t)\| \le \frac{3+N}{4}K \le K e^{\left[t-\left(\frac{(j-1)t_1}{M^{\frac{2}{a}}}\right)\right]^{a_1}},\ t\in\left[0,\frac{t_1}{M^{\frac{2}{a}}}\right] \qquad (2.7)$$

$$\|\Delta y(t)\| \le \|y_0\| + NK\left[1+\frac{2t_1^{a_1}}{M^{\frac{2a_1}{a}}}\right] + H + \frac{ht_1^{a}}{\Gamma(a+1)} + \frac{nK}{\Gamma(a)}\left[\int_0^{\frac{(M-1)t_1}{M^{1+\frac{2}{a}}}} (t-u)^{a-1} e^{u^{a_1}} du\right.$$

$$+\int_{\frac{(M-1)t_1}{M^{1+\frac{2}{a}}}}^{\frac{t_1}{M^{\frac{2}{a}}}} \left(\frac{t_1}{M^{\frac{2}{a}}}-u\right)^{a-1} e^{\left(\frac{t_1}{M^{\frac{2}{a}}}\right)^{a_1}} du + \int_{\frac{t_1}{M^{\frac{2}{a}}}}^{\frac{t_1}{M^{\frac{2}{a}}}+\frac{M-1}{M}\left(t-\frac{t_1}{M^{\frac{2}{a}}}\right)} (t-u)^{a-1} e^{\left[u-\frac{t_1}{M^{\frac{2}{a}}}\right]^{a_1}} du$$

$$\left.+\int_{\frac{t_1}{M^{\frac{2}{a}}}+\frac{M-1}{M}\left(t-\frac{t_1}{M^{\frac{2}{a}}}\right)}^{t} (t-u)^{a-1} e^{\left[\frac{t_1}{M^{\frac{2}{a}}}\right]^{a_1}} du\right]$$

$$\|\Delta y(t)\| \le \|y_0\| + NK\left[1+\frac{2t_1^{a_1}}{M^{a_1}}\right] + H + \frac{ht_1^{a}}{\Gamma(a+1)} + \frac{nK}{\Gamma(a)}\left[t_1^{a-a_1} M^{-2+\frac{2a_1}{a}} \int_0^{\frac{t_1}{M^{\frac{2}{a}}}} u^{a_1-1} e^{u^{a_1}} du\right.$$

$$+ \int_{\frac{t_1}{M^{\frac{2}{a}}}}^{\frac{2t_1}{M^{\frac{2}{a}}}-\frac{t_1}{M^{1+\frac{2}{a}}}} (t-u)^{a-1} e^{\left(u-\frac{t_1}{M^{\frac{2}{a}}}\right)^{a_1}} du + \int_{\frac{2t_1}{M^{\frac{2}{a}}}-\frac{t_1}{M^{1+\frac{2}{a}}}}^{\frac{2t_1}{M^{\frac{2}{a}}}} \left[\frac{2t_1}{M^{\frac{2}{a}}}-u\right]^{a-1} e^{\left(\frac{t_1}{M^{\frac{2}{a}}}\right)^{a_1}} du$$

$$+ \int_{\frac{2t_1}{M^{\frac{2}{a}}}}^{\frac{2t_1}{M^{\frac{2}{a}}}+\frac{M-1}{M}\left(t-\frac{2t_1}{M^{\frac{2}{a}}}\right)} \left(t-u\right)^{a-1} e^{\left[u-\frac{2t_1}{M^{\frac{2}{a}}}\right]^{a_1}} du + \int_{\frac{2t_1}{M^{\frac{2}{a}}}+\frac{M-1}{M}\left(t-\frac{2t_1}{M^{\frac{2}{a}}}\right)}^{t} \left(t-u\right)^{a-1} e^{\left(\frac{t_1}{M^{\frac{2}{a}}}\right)^{a_1}} du$$

$$\|\Delta y(t)\| \le \|y_0\| + NK\left[1+\frac{2t_1^{a_1}}{M^{\frac{2a_1}{a}}}\right] + H + \frac{ht_1^{a}}{\Gamma(a+1)} + \frac{2nt_1^{a} K\left[1+\frac{2t_1 a_1}{M^{\frac{2a_1}{a}}}\right]}{M^{2+a}\,\Gamma\left(1+a\right)}$$

$$+\left[\frac{nK}{\Gamma(a)}\left[\int_0^{\frac{(M-1)t_1}{M^{1+\frac{2}{a}}}} \left(\frac{t_1}{M^{\frac{2}{a}}}-u\right)^{a-1} e^{u^{a_1}} du + \int_{\frac{t_1}{M^{\frac{2}{a}}}}^{\frac{t_1}{M^{\frac{2}{a}}}+\frac{M-1}{M}\left(t-\frac{t_1}{M^{\frac{2}{a}}}\right)} \left[t-\frac{t_1}{M^{\frac{2}{a}}}-\left(u-\frac{t_1}{M^{\frac{2}{a}}}\right)\right]^{a-1} e^{\left[u-\frac{t_1}{M^{\frac{2}{a}}}\right]^{a_1}} du\right]\right]$$

$$\le K\left[\frac{1+N}{2}+\frac{2nt_1^{a}}{a_1 M^{1+a_1}\Gamma(a)}\right] + \frac{nK}{\Gamma(a)} \int_{\frac{t_1}{M^{\frac{2}{a}}}}^{\frac{t_1}{M^{\frac{2}{a}}}+\frac{M-1}{M}\left(t-\frac{t_1}{M^{\frac{2}{a}}}\right)} \left[t-\frac{t_1}{M^{\frac{2}{a}}}-\left(u-\frac{t_1}{M^{\frac{2}{a}}}\right)\right]^{a-1} e^{\left[u-\frac{t_1}{M^{\frac{2}{a}}}\right]^{a_1}} du$$

$$\le K\left[\frac{1+N}{2}+\frac{4nt_1^{a}}{a_1 M^{1+a_1}\Gamma(a)}\right]$$

$$\|\Delta y(t)\| \le \frac{3+N}{4} K \le K e^{\left[t-\left(\frac{(j-1)t_1}{M^{\frac{2}{a}}}\right)\right]^{a_1}},\ t\in\left[\frac{t_1}{M^{\frac{2}{a}}},\frac{2t_1}{M^{\frac{2}{a}}}\right] \quad (2.8)$$

$$\left\|\Delta y(t)\right\| \le \frac{1+N}{2}K + \frac{nK}{\Gamma(a)}\Bigg[t_1^{a-a_1} M^{-2+\frac{2a_1}{a}} \int_0^{\frac{t_1}{M^{\frac{2}{a}}}} u^{a_1-1} e^{u^{a_1}} du$$

$$+ \int_{\frac{t_1}{M^{\frac{2}{a}}}}^{\frac{2t_1}{M^{\frac{2}{a}}} - \frac{t_1}{M^{1+\frac{2}{a}}}} \left(\frac{2t_1}{M^{\frac{2}{a}}} - u\right)^{a-1} e^{\left(u - \frac{t_1}{M^{\frac{2}{a}}}\right)^{a_1}} du + \int_{\frac{2t_1}{M^{\frac{2}{a}}}}^{\frac{2t_1}{M^{\frac{2}{a}}} + \frac{M-1}{M}\left(t - \frac{2t_1}{M^{\frac{2}{a}}}\right)} \left(t-u\right)^{a-1} e^{\left(u - \frac{2t_1}{M^{\frac{2}{a}}}\right)^{a_1}} du \Bigg]$$

$$\le K\left[\frac{1+N}{2} + \frac{6n t_1^{a}}{a_1 M^{1+a_1}\Gamma(a)}\right] \le \frac{3+N}{4}K$$

$$\le K e^{\left(t - \frac{2t_1}{M^{\frac{2}{a}}}\right)^{a_1}}, \quad t \in \left[\frac{2t_1}{M^{\frac{2}{a}}}, \frac{3t_1}{M^{\frac{2}{a}}}\right], \tag{2.9}$$

$$\left\|\Delta y(t)\right\| \le \frac{1+N}{2}K + \frac{nK}{\Gamma(a)}\Bigg[2^{a_1-1} a_1^{a-a_1} M^{\frac{2a_1}{a-2}} \int_0^{\frac{t_1}{M^{\frac{2}{a}}}} u^{a_1-1} e^{u^{a_1}} du$$

$$+ t_1^{a-a_1} M^{-2+\frac{2a_1}{a}} \int_{\frac{t_1}{M^{\frac{2}{a}}}}^{\frac{2t_1}{M^{\frac{2}{a}}}} \left(u - \frac{t_1}{M^{\frac{2}{a}}}\right)^{a_1-1} e^{\left(u - \frac{t_1}{M^{\frac{2}{a}}}\right)^{a_1}} du + \int_{\frac{2t_1}{M^{\frac{2}{a}}}}^{\frac{3t_1}{M^{\frac{2}{a}}} - \frac{t_1}{M^{1-\frac{2}{a}}}} \left(\frac{3t_1}{M^{\frac{2}{a}}} - u\right)^{a-1} e^{\left(u - \frac{2t_1}{M^{\frac{2}{a}}}\right)^{a_1}} du$$

$$+ \int_{\frac{3t_1}{M^{\frac{2}{a}}}}^{\frac{3t_1}{M^{\frac{2}{a}}} + \frac{M-1}{M}\left(t - \frac{3t_1}{M^{\frac{2}{a}}}\right)} \left(t-u\right)^{a-1} e^{\left(u - \frac{3t_1}{M^{\frac{2}{a}}}\right)^{a_1}} du \Bigg] \le K\left[\frac{1+N}{2} + \frac{6n t_1^{a}}{a_1 M^{1+a_1}\Gamma(a)} + \frac{2n M^{-2} t_1^{a}}{a_1 \Gamma(a)} 2^{a_1-1}\right]$$

$$\le \frac{3+N}{4}K \le K e^{\left(t - \frac{3t_1}{M^{\frac{2}{a}}}\right)^{a_1}} \tag{2.10}$$

for $t \in \left[\frac{3t_1}{M^{\frac{2}{a}}}, \frac{4t_1}{M^{\frac{2}{a}}} \right]$,.........

$$\|\Delta y(t)\| \le \frac{1+N}{2}K + \frac{nK}{\Gamma(a)} \left\{ t_1^{a-a_1} M^{\frac{2a_1}{a}-2} \left[(j-2)^{a_1-1} \int_0^{\frac{t_1}{M^{\frac{2}{a}}}} u^{a_1-1} e^{u^{a_1}} du + (j-3)^{a_1-1} \int_{\frac{t_1}{M^{\frac{2}{a}}}}^{\frac{2t_1}{M^{\frac{2}{a}}}} u^{a_1-1} e^{u^{a_1}} du \right.\right.$$

$$\left. + ... + 2^{a_1-1} \int_{\frac{(j-4)t_1}{M^{\frac{2}{a}}}}^{\frac{(j-3)t_1}{M^{\frac{2}{a}}}} u^{a_1-1} e^{u^{a_1}} du \right] + t_1^{a-a_1} M^{-2+\frac{2a_1}{a}} \int_{\frac{(j-3)t_1}{M^{\frac{2}{a}}}}^{\frac{(j-2)t_1}{M^{\frac{2}{a}}}} \left(u - \frac{(j-3)t_1}{M^{\frac{2}{a}}} \right)^{a_1-1} e^{\left(u - \frac{(j-3)t_1}{M^{\frac{2}{a}}} \right)^{a_1}} du$$

$$\left. + \int_{\frac{(j-2)t_1}{M^{\frac{2}{a}}}}^{\frac{(j-1)t_1}{M^{\frac{2}{a}}} - \frac{t_1}{M^{1+\frac{2}{a}}}} \left(\frac{(j-1)t_1}{M^{\frac{2}{a}}} - u \right)^{a-1} e^{\left(u - \frac{(j-2)t_1}{M^{\frac{2}{a}}} \right)^{a_1}} du + \int_{\frac{(j-1)t_1}{M^{\frac{2}{a}}}}^{\frac{(j-1)t_1}{M^{\frac{2}{a}}} + \frac{M-1}{M}\left(t - \frac{(j-1)t_1}{M^{\frac{2}{a}}} \right)} (t-u)^{a-1} e^{\left(u - \frac{(j-1)t_1}{M^{\frac{2}{a}}} \right)^{a_1}} du \right\}$$

$$\|\Delta y(t)\| \le K \left[\frac{1+N}{2} + \frac{6nt_1^a}{a_1 M^{1+a_1}\Gamma(a)} + \frac{2nM^{-2} t_1^a}{a_1 \Gamma(a)} \sum_{i=2}^{j-2} i^{a_1-1} \right]$$

$$\le K \left[\frac{1+N}{2} + \frac{6nt_1^a}{a_1 M^{1+a_1}\Gamma(a)} + \frac{2nM^{-2} t_1^a}{a_1 \Gamma(a)} \sum_{i=2}^{M^{\frac{2}{a}}-2} i^{a_1-1} \right]$$

$$\le K \left[\frac{1+N}{2} + \frac{6nt_1^a}{a_1 M^{1+a_1}\Gamma(a)} + \frac{2nM^{-2\left[1-\frac{a_1}{a}\right]} t_1^a}{a_1^2 \Gamma(a)} \right]$$

$$\|\Delta y(t)\| \le K\, e^{\left(t - \frac{(j-1)t_1}{M^{\frac{2}{a}}} \right)^{a_1}}, \tag{2.11}$$

for $t \in \left[\frac{(j-1)t_1}{M^{\frac{2}{a}}}, \frac{jt_1}{M^{\frac{2}{a}}} \right]$ (or) $t \in \left[\frac{(M^{\frac{2}{a}})t_1}{M^{\frac{2}{a}}}, t_1 \right]$ and

$\left(M^{\frac{2}{a}} \right) + 1 \ge j \ge 5$, which together with (I) and (2.7)-(2.10) yields that $\Delta : D \to D$.

STEP: 2

From (A1) and (B1), it follows that, for any $\varepsilon > 0$, $y, x \in D$, there exist $\rho(\varepsilon) > 0$ such that if $\|y - x\| < \rho$, then

$$\| f(x) - f(y) \| + \sup_{t \in L} \| g(t, y(t)) - g(t, x(t)) \| \le \varepsilon$$

The above equation together with (2.4) and (2.6) yields

$$\begin{aligned}
\| \Delta(y(t) - x(t)) \| &= \left\| f(x) - f(y) + \frac{1}{\Gamma(a)} \int_0^t (t-u)^{a-1} [g(u, y(u)) - g(u, x(u))] du \right\| \\
&\le \| f(x) - f(y) \| + \frac{1}{\Gamma(a)} \int_0^t (t-u)^{a-1} \| g(u, y(u)) - g(u, x(u)) \| du \\
&\le \| f(x) - f(y) \| + \frac{\sup_{t \in L} \| g(t, y(t)) - g(t, x(t)) \|}{\Gamma(a)} \int_0^t (t-u)^{a-1} du \\
&\le 2 \max \left\{ 1, \frac{t_1^{\,a}}{\Gamma(a+1)} \right\} \left[\| f(x) - f(y) \| + \sup_{t \in L} \| g(t, y(t)) - g(t, x(t)) \| \right] \\
&\le 2 \max \left\{ 1, \frac{t_1^{\,a}}{\Gamma(a+1)} \right\} \varepsilon
\end{aligned}$$

Hence the mapping Δ is continuous.

STEP: 3

To prove Δ is equicontinuous.

Let $t_2, t_3 \in L$ with $t_2 < t_3$ and $y \in D$. Then, from (A1), (2.4) and (2.6), we obtain

$$\begin{aligned}
\| \Delta y(t_3) - \Delta y(t_2) \| &\le \left\| \int_0^{t_3} \frac{(t_3 - u)^{a-1}}{\Gamma(a)} g(u, y(u)) du - \int_0^{t_2} \frac{(t_2 - u)^{a-1}}{\Gamma(a)} g(u, y(u)) du \right\| \\
&\le \int_{t_2}^{t_3} \frac{(t_3 - u)^{a-1}}{\Gamma(a)} \| g(u, y(u)) \| du + \int_0^{t_2} \frac{(t_2 - u)^{a-1} - (t_3 - u)^{a-1}}{\Gamma(a)} \| g(u, y(u)) \| du \\
&\le (2nK + h) \left[\int_{t_2}^{t_3} \frac{(t_3 - u)^{a-1}}{\Gamma(a)} du + \int_0^{t_2} \frac{(t_2 - u)^{a-1} - (t_3 - u)^{a-1}}{\Gamma(a)} du \right] \\
&\le \frac{2(2nK + h)}{\Gamma(1+a)} (t_3 - t_2)^a
\end{aligned}$$

Hence Δ is equicontinuous.

By Arzela-Ascoli theorem and the above steps 1-3, it can be concluded that $\Delta : D \to D$ is completely continuous.

From (2.1) and Schauder fixed point theorem, the problem (1.1) there exists at least one solution in D for the problem (1.1).

Hence the proof.

Theorem 1.2

Under the condition (A2), (B2) and (I), (1.1) has unique solution.

Proof

By known conditions (A2) and (B2), it can be easily seen that required.

If $h = \|g(t,0)\|_0$ and $H = f(0)$, then the criteria of theorem (1.1) have been satisfied.

Hence, by conditions of theorem (1.2), there exist at least one solution for (1.1).

Let us clarify the uniqueness [6]-[9].

Assume that (1.1) has two solutions

$$y \text{ and } x \text{ with } \|y - x\|_0 > 0. \tag{2.12}$$

and there exist $\eta \in \left[\frac{(j-1)t_1}{M^{\frac{2}{a}}}, \frac{jt_1}{M^{\frac{2}{a}}} \right]$ where M, j are as in (2.1) and

$$\|y(\eta) - x(\eta)\| > K\, e^{\left[\eta - \frac{(j-1)t_1}{M^{\frac{2}{a}}} \right]^{a_1}} \tag{2.13}$$

$$where\ K = \frac{1+N}{2} \|y - x\|_0$$

$$\text{Set } \xi = \min \left\{ t \in \left[\frac{(j-1)t_1}{M^{\frac{2}{a}}}, \frac{jt_1}{M^{\frac{2}{a}}} \right] \left(j \in \left\{ 1, 2, \ldots\ldots. \left[M^{\frac{2}{a}} \right] \right\} \right) \right.$$

$$\left. \text{or } t \in \left[\frac{\left(M^{\frac{2}{a}} \right) t_1}{M^{\frac{2}{a}}}, t_1 \right] : \times \|y(t) - x(t)\| > K e^{\left[t - \frac{(j-1)t_1}{M^{\frac{2}{a}}} \right]^{a_1}} \right\}$$

Then from (B2) and (2.13), hence the result is

$$\text{for } 0 < \xi < \eta,\ \|y(\xi) - x(\xi)\| = K\, e^{\left[\xi - \frac{(j-1)t_1}{M^{\frac{2}{a}}} \right]^{a_1}}$$

similarly from (2.7)-(2.11), it is obtained

$$\|y(\xi) - x(\xi)\| \le \frac{3+N}{4} K$$

by (2.16), the above is contradiction.

Hence $\|y-x\|_0 = 0.$

Hence the proof.

Example 1.1

Consider the problem

$$D^{\frac{1}{2}}y(t) = t - 2\sqrt{\|y(t)\|},\ t \in L := [0, t_1]$$
$$y(0) + \frac{\|y\|}{2} = \frac{1}{4}. \qquad (3.1)$$

Solution

Given $a = \frac{1}{2}, N = \frac{1}{2}, n = \frac{1}{2}$ and $h = 1 + t_1$

condition (I′) $\Rightarrow \quad N + \frac{n t_1{}^a}{\Gamma(1+a)} < 1$ (for given information)

Hence (I′) is satisfied. But if theorem(1.1) holds and from $1+2\|y\| \geq 2\sqrt{\|y(t)\|}$, example (3.1) has at least one solution.

Example 1.2

Consider the problem

$$D^{\frac{1}{2}}y(t) = t - 10y(t),\ t \in L := [0, t_1]$$
$$y(0) + \frac{\|y\|}{2} = \frac{1}{4}. \qquad (3.2)$$

Solution

Given $a = \frac{1}{2}, N = \frac{1}{2},\ n = 10$

condition (I′) $\Rightarrow N + \frac{n t_1{}^a}{\Gamma(1+a)} < 1$ satisfied.

If theorem (1.2) holds, then there exists a unique solution([8]-[9]) for problem (3.2) [which is new result].

2. Conclusion

From the above discussion, a new attempt has been made to prove existence of unique solution for global Cauchy problem by fixing one new condition with proper example.

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