

Oscillation of Even order Impulsive Partial Differential Equations with Deviating Arguments

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Abstract

In this paper, we consider a boundary value problem associated with even order impulsive delay partial differential equations. Sufficient conditions are obtained for the oscillation of solutions using impulsive differential inequalities and integral averaging scheme with Dirichlet boundary condition. Examples are provided to illustrate the obtained results.

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1 Introduction

The oscillation of parabolic, hyperbolic and elliptic equations has been extensively studied in the literature. The initial work on impulsive partial differential equation [5] was published in 1991. The problem of obtaining sufficient conditions to ensure that oscillation of solutions of certain classes of higher order partial differential equations has been studied by a number of authors, see [6, 7, 8, 16], and references therein. In recent years, the theory of impulsive differential equations with and without delay has become more important in real life applications. For example, jerk equations [9], sliding and oscillations in fisheries [3], respiratory model [11], traffic flow [12], podpulation ecology, climate model [17], epidemic model [4], human respiratory stability [13], tumor growth [14], beam equations [18]. The results in this paper expand and improve numerous findings in the earlier publications not including impulse. In the case $m = 2$, the results are reduced to the second order non-impulsive partial differential equation and which was studied in [2]. We anticipate that this work acquire the concentration of numerous researchers functioning on the even order impulsive partial functional differential equations.

In this paper, we are concerned with the oscillatory behavior of solutions of the even order impulsive partial differential equation of the form

$$\left. \begin{aligned} \frac{\partial^m}{\partial t^m} u(x, t) + \sum_{j=1}^l Q_j(x, t) u(x, \rho_j(t)) &= a(t) \Delta u(x, t) \\ &+ \sum_{s=1}^n p_s(t) \Delta u(x, \tau_s(t)), \quad t \neq t_k, \quad (x, t) \in \Omega \times (0, +\infty) \equiv G, \\ \frac{\partial^{(i)} u(x, t_k^+)}{\partial t^{(i)}} &= (1 + \alpha_k^{(i)}) \frac{\partial^{(i)} u(x, t_k)}{\partial t^{(i)}}, \\ &k = 1, 2, \dots, \quad i = 0, 1, 2, \dots, m-1 \end{aligned} \right\} \quad (1)$$

with the boundary condition

$$u = 0, \quad (x, t) \in \partial\Omega \times (0, +\infty), \quad (2)$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$ and Δ is the Laplacian in the Euclidean space $\mathbb{R}^N$.

In this paper, we assume that the following hypotheses hold:

(H₁) $a(t), p_s(t) \in PC([0, +\infty), [0, +\infty))$, $s = 1, 2, \dots, n$, where PC represents the class of functions which are piecewise continuous in t with discontinuities of first kind only at $t = t_k$, $k = 1, 2, \dots$, and left continuous at $t = t_k$, $k = 1, 2, \dots$.

(H₂) $Q_j(x, t) \in C(\bar{\Omega} \times [0, +\infty), [0, +\infty))$, $q_j(t) = \min_{x \in \bar{\Omega}} Q_j(x, t)$,
 $\rho_j(t) \in C([0, +\infty), \mathbb{R})$, $\lim_{t \rightarrow +\infty} \rho_j(t) = +\infty$, $j = 1, 2, \dots, l$,
 $\tau_s(t) \in C([0, +\infty), \mathbb{R})$, $\lim_{t \rightarrow +\infty} \tau_s(t) = +\infty$, $s = 1, 2, \dots, n$.

(H₃) $\frac{\partial^{(i)} u(x, t)}{\partial t^{(i)}}$ are piecewise continuous in t with discontinuities of first kind only at $t = t_k$, $k = 1, 2, \dots$, and left continuous at $t = t_k$, $\frac{\partial^{(i)} u(x, t_k)}{\partial t^{(i)}} = \frac{\partial^{(i)} u(x, t_k^-)}{\partial t^{(i)}}$, $k = 1, 2, \dots$, $i = 0, 1, 2, \dots, m-1$.

(H₄) $\alpha_k^{(i)} > -1$, $\alpha_k^{(0)} < \alpha_k^{(m-1)}$, $0 < t_1 < \dots < t_k < \dots$ and $\lim_{t \rightarrow +\infty} t_k = +\infty$, $k = 1, 2, \dots$.

2 Preliminaries

In this section, we begin with definitions and well known results which are required throughout this paper.

Definition 1. A solution u of the problem (1) is a function $u \in C^m(\bar{\Omega} \times [t_{-1}, +\infty), \mathbb{R}) \cap C(\bar{\Omega} \times [\hat{t}_{-1}, +\infty), \mathbb{R})$ that satisfies (1), where

$$t_{-1} := \min \left\{ 0, \min_{1 \leq j \leq l} \left\{ \inf_{t \geq 0} \rho_j(t) \right\} \right\}, \quad \hat{t}_{-1} := \min \left\{ 0, \min_{1 \leq s \leq n} \left\{ \inf_{t \geq 0} \tau_s(t) \right\} \right\}.$$

Definition 2. The solution u of the problem (1),(2) is said to be oscillatory in the domain G if for any positive number ℓ there exists a point $(x_0, t_0) \in \Omega \times [\ell, +\infty)$ such that $u(x_0, t_0) = 0$ holds.

Definition 3. A function $V(t)$ is said to be eventually positive (negative), if there exists a $t_1 \geq t_0$ such that $V(t) > 0$ (< 0) holds for all $t \geq t_1$.

It is identified that [15] the least eigenvalue $\beta_0 > 0$ of the eigenvalue problem

$$\begin{aligned}\Delta\omega(x) + \beta\omega(x) &= 0, \text{ in } \Omega, \\ \omega(x) &= 0, \text{ on } \partial\Omega,\end{aligned}$$

and the consequent eigen function $\Phi(x) > 0$ in Ω .

For each positive solution $u(x, t)$ of the problem (1),(2) we combine the function $V(t)$ defined by

$$V(t) = K_\Phi \int_{\Omega} u(x, t) \Phi(x) dx,$$

where

$$K_\Phi = \left(\int_{\Omega} \Phi(x) dx \right)^{-1}, \quad |\Omega| = \int_{\Omega} dx.$$

Lemma 4. [10] Let $y(t) \in C^n([t_0, \infty), (0, +\infty))$. If the derivative $y^{(n)}(t)$ is eventually of one sign for t , then there exist a $t_x \geq t_0$ and an integer l , $(0 \leq l \leq n)$, with $n+l$ even for $y^{(n)}(t) \geq 0$, or $n+l$ odd for $y^{(n)}(t) \leq 0$ such that

$$\begin{aligned}l > 0 \text{ yields } y^{(k)}(t) > 0, \quad t \geq t_x, \quad k = 1, 2, \dots, l-1, \\ \text{and } l \leq n-1 \text{ yields } (-1)^{l+k} y^{(k)}(t) > 0, \quad t \geq t_x, \quad k = l, l+1, l+2, \dots, n-1.\end{aligned}$$

Lemma 5. [1] Let $y(t)$ be a positive and n times differentiable function on $[t_0, +\infty)$, $y^{(n)}(t)y^{(n-1)}(t) \leq 0$ for $t \geq t_*$, and assume that $\lim_{t \rightarrow \infty} y(t) \neq 0$. Then for every $\lambda \in (0, 1)$ there exists a $t_\lambda \in [t_*, \infty)$ such that, for all $t \in [t_\lambda, \infty)$,

$$y(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} \left| y^{(n-1)}(t) \right|.$$

3 Main Results

In this section, we establish some sufficient conditions for the oscillation of all solutions of the problem (1),(2).

Theorem 6. Suppose that conditions $(H_1) - (H_4)$ hold and that every solution $u(x, t)$ of the problem (1),(2) is oscillatory in G , if the impulsive delay differential inequality

$$\left. \begin{aligned} V^{(m)}(t) + \sum_{j=1}^l q_j(t) V(\rho_j(t)) &\leq 0, \quad t \neq t_k, \quad t \geq t_1, \\ \frac{\partial^{(i)} V(t_k^+)}{\partial t^{(i)}} &= \left(1 + \alpha_k^{(i)} \right) \frac{\partial^{(i)} V(t_k)}{\partial t^{(i)}}, \quad t = t_k, \\ &k = 1, 2, \dots, i = 0, 1, 2, \dots, m-1, \end{aligned} \right\} \quad (3)$$

has no eventually positive solutions.

Proof. Let $u(x, t)$ be a nonoscillatory solution of the boundary value problem (1), (2) and $u(x, t) > 0$. Then there exists a $t_1 > t_0 > 0$ such that $\rho_j(t) \geq t_0$, $\tau_s(t) \geq t_0$.

$$\begin{aligned}u(x, \rho_j(t)) &> 0, \quad \text{for } (x, t) \in \Omega \times [t_1, +\infty), \quad j = 1, 2, \dots, l, \\ u(x, \tau_s(t)) &> 0, \quad \text{for } (x, t) \in \Omega \times [t_1, +\infty), \quad s = 1, 2, \dots, n.\end{aligned}$$

Multiplying both sides of equation (1) by $K_\Phi \Phi(x) > 0$ and integrating with respect to x over Ω , we obtain

$$\frac{d^m}{dt^m} \left[\int_{\Omega} u(x, t) K_\Phi \Phi(x) dx \right] + \sum_{j=1}^l Q_j(x, t) \int_{\Omega} u(x, \rho_j(t)) K_\Phi \Phi(x) dx$$

$$= a(t) \int_{\Omega} \Delta u(x, t) K_{\Phi} \Phi(x) dx + \sum_{s=1}^n p_s(t) \int_{\Omega} \Delta u(x, \tau_s(t)) K_{\Phi} \Phi(x) dx. \quad (4)$$

From Green's formula and the boundary condition (2), we see that

$$\begin{aligned} K_{\Phi} \int_{\Omega} \Delta u(x, t) \Phi(x) dx &= K_{\Phi} \int_{\partial\Omega} \left[\Phi(x) \frac{\partial u}{\partial \gamma} - u \frac{\partial \Phi(x)}{\partial \gamma} \right] dS + K_{\Phi} \int_{\Omega} u(x, t) \Delta \Phi(x) dx \\ &= -\beta_0 V(t). \\ \therefore K_{\Phi} \int_{\Omega} \Delta u(x, t) \Phi(x) dx &\leq 0, \end{aligned} \quad (5)$$

and for $s = 1, 2, \dots, n$, we have

$$\begin{aligned} K_{\Phi} \int_{\Omega} \Delta u(x, \tau_s(t)) \Phi(x) dx &= K_{\Phi} \int_{\partial\Omega} \left[\Phi(x) \frac{\partial u(x, \tau_s(t))}{\partial \gamma} - u(x, \tau_s(t)) \frac{\partial \Phi(x)}{\partial \gamma} \right] dS \\ &\quad + K_{\Phi} \int_{\Omega} u(x, \tau_s(t)) \Delta \Phi(x) dx \\ &= -\beta_0 V(\tau_s(t)), \\ \therefore K_{\Phi} \int_{\Omega} \Delta u(x, \tau_s(t)) \Phi(x) dx &\leq 0. \end{aligned} \quad (6)$$

Where dS is surface component on $\partial\Omega$. Combining (4)-(6), we get that

$$\frac{d^m V(t)}{dt^m} + \sum_{j=1}^l q_j(t) V(\rho_j(t)) \leq 0, \quad t \neq t_k, \quad t \geq t_1. \quad (7)$$

Also, multiplying both sides of the equation (1) by $K_{\Phi} \Phi(x) > 0$, integrating with respect to x over Ω , and from (H_4) we obtain

$$\begin{aligned} \frac{\partial^{(i)}}{\partial t^{(i)}} K_{\Phi} \int_{\Omega} u(x, t_k^+) \Phi(x) dx &= (1 + \alpha_k^{(i)}) \frac{\partial^{(i)}}{\partial t^{(i)}} K_{\Phi} \int_{\Omega} u(x, t_k) \Phi(x) dx, \\ \frac{\partial^{(i)} V(t_k^+)}{\partial t^{(i)}} &= (1 + \alpha_k^{(i)}) \frac{\partial^{(i)} V(t_k)}{\partial t^{(i)}}. \end{aligned} \quad (8)$$

Therefore $V(t)$ is an eventually positive solution of (3). This disagree with the hypothesis and completes the proof of the theorem. $\square$

Theorem 7. Suppose that conditions (H_1) - (H_4) hold and that

$$\rho(t) = \max_{1 \leq j \leq l} \rho_j(t) \leq t, \quad \rho'(t) \geq 0.$$

If for every number $\lambda_0 \in (0, 1)$,

$$\limsup_{t \rightarrow \infty} \frac{1}{(m-1)!} \int_{\rho(t)}^t \sum_{j=1}^l \prod_{t_0 \leq t_k < s} \left(\frac{1 + \alpha_k^{(m-1)}}{1 + \alpha_k^{(0)}} \right) q_j(s) \rho_j^{m-1}(s) ds > \frac{1}{\lambda_0}, \quad (9)$$

then every solution $u(x, t)$ of the problem (1), (2) is oscillatory in G .

Proof. On the contrary, let $u(x, t)$ be a nonoscillatory solution of the boundary value problem (1), (2) which we assume to be positive. Now we use

$$V^{(m)}(t) + \sum_{j=1}^l q_j(t) V(\rho_j(t)) \leq 0, \quad t \geq t_1.$$

By Lemma 5

$$V^{(m)}(t) + \sum_{j=1}^l q_j(t) \frac{\lambda_0}{(m-1)!} \rho_j^{m-1}(t) V^{(m-1)}(\rho_j(t)) \leq 0, \quad t \geq t_1. \quad (10)$$

Define

$$W(t) = \prod_{t_0 \leq t_k < t} \left(\frac{1 + \alpha_k^{(m-1)}}{1 + \alpha_k^{(0)}} \right)^{-1} V(t).$$

In fact, $W(t)$ is continuous on each interval $(t_k, t_{k+1}]$, and in consideration of $W(t_k^+) \leq \left(\frac{1 + \alpha_k^{(m-1)}}{1 + \alpha_k^{(0)}} \right) W(t_k)$, it follows for $t \geq t_0$ that, inequality (10) has no eventually positive solution, if the following (11) has no eventually positive solution.

$$W^{(m)}(t) + \sum_{j=1}^l q_j(t) \frac{\lambda_0}{(m-1)!} \rho_j^{m-1}(t) W^{(m-1)}(\rho_j(t)) \leq 0, \quad t \geq t_1. \quad (11)$$

Where

$$W(t_k^+) = \prod_{t_0 \leq t_k \leq t_n} \left(\frac{1 + \alpha_k^{(m-1)}}{1 + \alpha_k^{(0)}} \right)^{-1} V(t_k^+) = \prod_{t_0 \leq t_k < t_n} \left(\frac{1 + \alpha_k^{(m-1)}}{1 + \alpha_k^{(0)}} \right)^{-1} V(t_k) = W(t_k),$$

$$W(t_k^-) = \prod_{t_0 \leq t_k \leq t_{n-1}} \left(\frac{1 + \alpha_k^{(m-1)}}{1 + \alpha_k^{(0)}} \right)^{-1} V(t_k^-) = \prod_{t_0 \leq t_k < t_n} \left(\frac{1 + \alpha_k^{(m-1)}}{1 + \alpha_k^{(0)}} \right)^{-1} V(t_k) = W(t_k).$$

Which implies that, $W(t)$ is continuous on $[t_0, +\infty)$. Then we get

$$\prod_{t_0 \leq t_k < t} \left(\frac{1 + \alpha_k^{(m-1)}}{1 + \alpha_k^{(0)}} \right)^{-m} V^{(m)}(t) + \sum_{j=1}^l q_j(t) \frac{\lambda_0}{(m-1)!} \rho_j^{m-1}(t) \prod_{t_0 \leq t_k < t} \left(\frac{1 + \alpha_k^{(m-1)}}{1 + \alpha_k^{(0)}} \right)^{-m+1} V^{(m-1)}(\rho_j(t)) \leq 0.$$

Integrating the above inequality from $\rho(t)$ to t , we have

$$\int_{\rho(t)}^t \prod_{t_0 \leq t_k < s} \left(\frac{1 + \alpha_k^{(m-1)}}{1 + \alpha_k^{(0)}} \right)^{-m} V^{(m)}(s) ds + \frac{\lambda_0}{(m-1)!} \int_{\rho(t)}^t \sum_{j=1}^l q_j(s) \rho_j^{m-1}(s) \prod_{t_0 \leq t_k < s} \left(\frac{1 + \alpha_k^{(m-1)}}{1 + \alpha_k^{(0)}} \right)^{-m+1} V^{(m-1)}(\rho_j(s)) ds \leq 0, \quad t \geq t_1.$$

From this,

$$\prod_{t_0 \leq t_k < t} \left(\frac{1 + \alpha_k^{(m-1)}}{1 + \alpha_k^{(0)}} \right)^{-m} [V^{(m-1)}(t) - V^{(m-1)}(\rho(t))] + \frac{\lambda_0}{(m-1)!} \int_{\rho(t)}^t \sum_{j=1}^l \prod_{t_0 \leq t_k < s} \left(\frac{1 + \alpha_k^{(m-1)}}{1 + \alpha_k^{(0)}} \right)^{-m+1} q_j(s) \rho_j^{m-1}(s) V^{(m-1)}(\rho_j(s)) ds \leq 0.$$

Divided by $\prod_{t_0 \leq t_k < t} \left(\frac{1 + \alpha_k^{(m-1)}}{1 + \alpha_k^{(0)}} \right)^{-m} V^{(m-1)}(\rho(t))$ on both sides, it follows that

$$\begin{aligned} \frac{V^{(m-1)}(t)}{V^{(m-1)}(\rho(t))} - 1 + \frac{\lambda_0}{(m-1)!} \int_{\rho(t)}^t \sum_{j=1}^l \prod_{t_0 \leq t_k < s} \left(\frac{1 + \alpha_k^{(m-1)}}{1 + \alpha_k^{(0)}} \right) q_j(s) \rho_j^{m-1}(s) ds &\leq 0, \\ \frac{\lambda_0}{(m-1)!} \int_{\rho(t)}^t \sum_{j=1}^l \prod_{t_0 \leq t_k < s} \left(\frac{1 + \alpha_k^{(m-1)}}{1 + \alpha_k^{(0)}} \right) q_j(s) \rho_j^{m-1}(s) ds &\leq 1 - \frac{V^{(m-1)}(t)}{V^{(m-1)}(\rho(t))}, \\ \frac{\lambda_0}{(m-1)!} \int_{\rho(t)}^t \sum_{j=1}^l \prod_{t_0 \leq t_k < s} \left(\frac{1 + \alpha_k^{(m-1)}}{1 + \alpha_k^{(0)}} \right) q_j(s) \rho_j^{m-1}(s) ds &\leq 1, \end{aligned}$$

and hence

$$\limsup_{t \rightarrow \infty} \frac{1}{(m-1)!} \int_{\rho(t)}^t \sum_{j=1}^l \prod_{t_0 \leq t_k < s} \left(\frac{1 + \alpha_k^{(m-1)}}{1 + \alpha_k^{(0)}} \right) q_j(s) \rho_j^{m-1}(s) ds \leq \frac{1}{\lambda_0}. \quad (12)$$

This violate the condition (9). The proof of the case $u(x, t) < 0$ is similar and is omitted. $\square$

4 Examples

In this section, we present couple of examples to point up our results established in Section 3.

Example 8. Consider the following equation

$$\left. \begin{aligned} \frac{\partial^4}{\partial t^4} u(x, t) + \frac{5}{3} u(x, t - 3\pi) &= \frac{4}{3} \Delta u(x, t) + \frac{2}{3} \Delta u(x, t - \pi), \\ &\quad t > 1, t \neq t_k, k = 1, 2, \dots, \\ u(x, t_k^+) &= \frac{k+1}{k} u(x, t_k), \\ \frac{\partial^{(i)}}{\partial t^{(i)}} u(x, t_k^+) &= \frac{\partial^{(i)}}{\partial t^{(i)}} u(x, t_k), \quad i = 1, 2, 3, \quad k = 1, 2, \dots, \end{aligned} \right\} \quad (13)$$

for $(x, t) \in (0, \pi) \times [0, +\infty)$, with the boundary condition

$$u(0, t) = u(\pi, t) = 0. \quad (14)$$

Here $\Omega = (0, \pi)$, $m = 4$, $n = 1$, $l = 1$, $\alpha_k^{(0)} = \frac{k+1}{k}$, $\alpha_k^{(i)} = 1$, $i = 1, 2, 3$,

$$\rho(t) = t - 3\pi, \quad \tau(t) = t - \pi, \quad q(t) = 5/3, \quad a(t) = 4/3, \quad p(t) = 2/3.$$

Also, we see from the above assumption that the hypotheses $(H_1) - (H_4)$ hold, moreover

$$\begin{aligned} \lim_{t \rightarrow +\infty} \int_{t_0}^t \prod_{t_0 \leq t_k < s} \left(\frac{1 + \alpha_k^{(m-1)}}{1 + \alpha_k^{(0)}} \right) ds &= \int_1^{+\infty} \prod_{1 < t_k < s} \left(\frac{1 + 1}{1 + \left(\frac{k+1}{k} \right)} \right) ds \\ &= \int_1^{t_1} \prod_{1 < t_k < s} \frac{2k}{2k+1} ds + \int_{t_1^+}^{t_2} \prod_{1 < t_k < s} \frac{2k}{2k+1} ds + \int_{t_2^+}^{t_3} \prod_{1 < t_k < s} \frac{2k}{2k+1} ds + \dots \end{aligned}$$

Now, the condition (9) reads,

$$\limsup_{t \rightarrow +\infty} \frac{5}{3} \int_{t-3\pi}^t \prod_{1 < t_k < s} \frac{2k}{2k+1} (s-3\pi)^3 ds = +\infty. \quad (15)$$

Therefore all the conditions of the Theorem 7 are satisfied. So, every solution of equations (13) & (14) is oscillatory in G . In fact, $u(x, t) = \sin x \sin t$ is such a solution.

Example 9. Consider the following equation of the form

$$\left. \begin{aligned} \frac{\partial^2}{\partial t^2} u(x, t) + \frac{1}{3} u(x, t - \pi/2) &= \Delta u(x, t) + \frac{1}{3} \Delta u(x, t - \frac{3\pi}{2}), \quad t \neq t_k, k = 1, 2, \dots, \\ u(x, t_k^+) &= \frac{k+1}{k} u(x, t_k), \\ \frac{\partial^{(i)}}{\partial t^{(i)}} u(x, t_k^+) &= \frac{\partial^{(i)}}{\partial t^{(i)}} u(x, t_k), \quad i = 1, \quad k = 1, 2, \dots, \end{aligned} \right\} \quad (16)$$

for $(x, t) \in (0, \pi) \times [0, +\infty)$, with the boundary condition

$$u(0, t) = u(\pi, t) = 0. \quad (17)$$

Here $\Omega = (0, \pi)$, $m = 2$, $n = 1$, $l = 1$, $\alpha_k^{(0)} = \frac{k+1}{k}$, $\alpha_k^{(i)} = 1$, $i = 1$,

$$\rho(t) = t - \pi/2, \quad \tau(t) = t - 3\pi/2, \quad q(t) = 1/3, \quad p(t) = 1/3, \quad a(t) = 1.$$

Since $t_0 = 1$, $t_k = 2^k$.

From the above assumptions it is easy to see that the hypotheses (H_1) - (H_4) hold. Still show that the condition is satisfied. In fact this condition (9) reads

$$\limsup_{t \rightarrow +\infty} \frac{1}{3} \int_{t-\pi/2}^t \prod_{1 < t_k < s} \frac{2k}{2k+1} (s - \pi/2) ds = +\infty. \quad (18)$$

Therefore all the conditions of the Theorem 7 are satisfied. So, every solution of equations (16) & (17) is oscillatory in G . Actually, $u(x, t) = \sin x \cos t$ is such a solution.

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