

OSCILLATION OF GENERALIZED QUASILINEAR DIFFERENCE EQUATIONS

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ABSTRACT. In this paper the authors establish conditions for the oscillatory behavior of solutions of generalized second order quasi linear difference equation.

1. INTRODUCTION

The theory of difference equations is based on the operator Δ defined as $\Delta u(k) = u(k+1) - u(k)$, $k \in \mathbb{N} = \{0, 1, 2, \dots\}$. Eventhough many authors [1], [8] have suggested Δ as

$$(D) \quad \Delta u(k) = u(k+\ell) - u(k), \quad \ell \in (0, \infty).$$

In [4], authors labelled the operator Δ defined by (D) as Δ_ℓ and by defining its inverse Δ_ℓ^{-1} , many interesting results in number theory were obtained [2, 4, 5].

Recently, there has been an increasing interest in the study of the oscillatory behaviors to the solutions of nonlinear and linear difference equations [9, 10].

This paper is concerned with the oscillatory behavior of generalized quasilinear difference equations of the forms

$$(1) \quad \Delta_\ell \left(a((k-1)\ell + j) |\Delta_\ell v((k-1)\ell + j)|^{\gamma-1} \Delta_\ell v((k-1)\ell + j) \right) + q(k\ell + j) f(v(k\ell + j)) = 0, \quad k \in (0, \infty), \ell \in (0, \infty)$$

where Δ_ℓ is the generalized difference operator defined by $\Delta_\ell v(k) = v(k+\ell) - v(k)$, $j = k - [\frac{k}{\ell}]\ell$, $a(k)$, $q(k)$ are real valued function, $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous, nondecreasing and $f(x) > 0$ for $x \neq 0$, $\gamma > 0$ and λ is a positive integer.

By a solution of equation (1) we mean a nontrivial function $v(k\ell + j)$ satisfying equation (1) for all $k\ell + j \geq 1$. A solution $v(k\ell + j)$ is said to be nonoscillatory if it is either eventually positive or eventually negative and osillatory otherwise.

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2. PRELIMINARIES

Definition 1. [3] For a real valued function $u(k)$, the generalized second order difference equation is defined by

$$(2) \quad v(k+2\ell) - 2v(k+\ell) + v(k) = u(k), \quad k \in [0, \infty), \ell \in (0, \infty).$$

Definition 2. [6] (Product Formula). Let $u(k)$ and $v(k)$ be any two real valued functions. Then

$$\Delta_\ell\{u(k)v(k)\} = u(k+\ell)\Delta_\ell v(k) + v(k)\Delta_\ell u(k)$$

The following are assumed through out this paper

- (i) $q(k\ell+j)$ is a nonnegative real valued function with infinitely many positive terms,
- (ii) $-\infty < \sum_{k=1}^{\infty} q(k\ell+j) < \infty$,
- (iii) $R(k\ell+j, k_0\ell+j) = \sum_{s=k_0}^{k-1} \frac{1}{a^{1/\gamma}(s\ell+j)}$, $R(k\ell+j, k_0\ell+j) \rightarrow \infty$ and $R(k\ell+j) = R(k\ell+j, 1)$,
- (iv) $\left| \frac{f(x)}{x^\gamma} \right| \geq M > 0$ for $|x| > \delta > 0$

Lemma 3. Assume that conditions (i) and (iii) hold. If $v(k\ell+j)$ is a solution of equation (1) such that $v(k\ell+j) > 0$ for $k\ell+j \geq K\ell+j \geq 1$ then $\Delta_\ell v(k\ell+j) > 0$ for $k\ell+j \geq K\ell+j$.

Proof. Since $a(k\ell+j)|\Delta_\ell v(k\ell+j)|^{\gamma-1}\Delta_\ell v(k\ell+j)$ is non increasing by equation (1), we see that $\Delta_\ell v(k\ell+j)$ is eventually of constant sign, that is $\Delta_\ell v(k\ell+j) > 0$ for $k\ell+j \geq K\ell+j \geq 1$ or there is $K_1\ell+j > K\ell+j$ such that $\Delta_\ell v(k\ell+j) < 0$ for $k\ell+j > K_1\ell+j$. If $\Delta_\ell v(k\ell+j) < 0$ for $k\ell+j > K_1\ell+j$ we have

$$(a(k\ell+j))^{\frac{1}{\gamma}}\Delta_\ell v(k\ell+j) \leq (a(K_2\ell+j))^{\frac{1}{\gamma}}\Delta_\ell v(K_2\ell+j)$$

for $k\ell+j \geq K_2\ell+j \geq K_1\ell+j$.

Dividing the last inequality by $(a(k\ell+j))^{\frac{1}{\gamma}}$ and summing from $K_2\ell+j$ to $k\ell+j$, we obtain

$$v((k+1)\ell+j) \leq v(K_2\ell+j) + a(K_2\ell+j)^{\frac{1}{\gamma}}\Delta_\ell v(K_2\ell+j) \sum_{s=K_2}^k \frac{1}{a(s\ell+j)^{1/\gamma}}.$$

As $(k\ell+j) \rightarrow \infty$ we see that $v(k\ell+j) \rightarrow \infty$, a contradiction. This completes the proof. $\square$

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Lemma 4. If $u, v \geq 0$ then $u^y - v^y \leq yu^{y-1}(u - v)$ for $y \geq 1$ and $u^y - v^y \leq yv^{y-1}(u - v)$ for $0 < y < 1$.

3. MAIN RESULTS

Theorem 5. Assume conditions (i) and (iii) hold. If

$$(3) \quad \liminf_{|x| \rightarrow \infty} \left| \frac{f(x)}{x^\gamma} \right| \geq d > 0,$$

there exists a positive real valued function $h(k\ell + j)$ such that

$$(4) \quad \lim_{k \rightarrow \infty} \sum_{s=1}^k h(s\ell + j)q(s\ell + j) = +\infty,$$

and

$$(5) \quad \lim_{k \rightarrow \infty} \sum_{s=1}^k \frac{a(s\ell + j)(\Delta_{\alpha(\ell)}h((s+1)\ell + j))^{\gamma+1}}{h^\gamma((s+1)\ell + j)} < +\infty,$$

then every solution of equation (1) is oscillatory.

Proof. Let $v(k\ell + j)$ be a nonoscillatory solution of equation (1), say $v(k\ell + j) > 0$ for $k \geq K \geq 1$, since the proof for the case $v(k\ell + j) < 0$ for $k \geq K$ is similar. Let

$$z((k-1)\ell + j) = \frac{a((k-1)\ell + j)(\Delta_\ell v((k-1)\ell + j))^\gamma}{v^\gamma((k-1)\ell + j)}, \quad k \geq K+l$$

since $\Delta_\ell v((k-1)\ell + j) > 0$ by Lemma 3. Then

$$(6) \quad \begin{aligned} \Delta_\ell(z((k-1)\ell + j)) &= -q(k\ell + j) \frac{f(v((k-l)\ell + j))}{v^\gamma((k-l)\ell + j)} \\ &- \frac{a(k\ell + j)(\Delta_\ell v(k\ell + j))^\gamma \Delta_\ell v^\gamma((k-l)\ell + j)}{v^\gamma((k+1-l)\ell + j)v^\gamma((k-l)\ell + j)} \end{aligned}$$

and hence

$$(7) \quad \begin{aligned} \Delta_\ell(z((k-1)\ell + j)) &+ \frac{z(k\ell + j)\Delta_\ell v^\gamma((k-l)\ell + j)}{v^\gamma((k-l)\ell + j)} \\ &+ q(k\ell + j) \frac{f(v((n-l)\ell + j))}{v^\gamma((n-l)\ell + j)} = 0, \quad \text{for } k \geq K+l. \end{aligned}$$

Since $\Delta_\ell v(k\ell + j) > 0$ for $k \geq K+l$, $\lim_{k \rightarrow \infty} v(k\ell + j)$ exists. If $\lim_{k \rightarrow \infty} v(k\ell + j) = b$, then $\lim_{k \rightarrow \infty} \frac{f(v((n-l)\ell + j))}{v^\gamma((n-l)\ell + j)} = \frac{f(b)}{b^\gamma} > 0$. If $\lim_{k \rightarrow \infty} v(k\ell + j) = b$, then

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$j) = +\infty$ then, in view of condition (3), $\liminf_{k \rightarrow \infty} \frac{f(v((n-l)\ell+j))}{v^\gamma((n-l)\ell+j)} = d_1 \geq d > 0$. Let $b^* = \min\left(\frac{d}{2}, \frac{f(b)}{b^\gamma}\right)$, from (7) we have

$$(8) \quad \Delta_\ell(z(k-1)\ell+j) + z(k\ell+j) \frac{\Delta_\ell v^\gamma((k-l)\ell+j)}{v^\gamma((k-l)\ell+j)} + b^*q(k\ell+j) \leq 0, \quad k \geq K+l.$$

From the mean value theorem, we have

$$(9) \quad \begin{aligned} & \Delta_\ell v^\gamma((k-l)\ell+j) \\ & \geq \begin{cases} \gamma v^{\gamma-1}((k-l)\ell+j) \Delta_\ell v((k-l)\ell+j), & \text{if } \gamma \geq 1 \\ \gamma v^{\gamma-1}((k+1-l)\ell+j) \Delta_\ell v((k-l)\ell+j), & \text{if } 0 < \gamma < 1 \end{cases} \end{aligned}$$

for $k \geq K+l$. Hence from (8) and (9), for $\gamma \geq 1$ we obtain

$$(10) \quad \Delta_\ell(z(k-1)\ell+j) + z(k\ell+j) \gamma \frac{\Delta_\ell v((k-l)\ell+j)}{v((k-l)\ell+j)} + b^*q(k\ell+j) \leq 0.$$

and for $0 < \gamma < 1$

$$(11) \quad \Delta_\ell(z(k-1)\ell+j) + b^*q(k\ell+j) + \left[\frac{\gamma z(k\ell+j)}{v^\gamma((k-l)\ell+j)} \times \Delta_\ell v^{\gamma-1}((k+1-l)\ell+j) \Delta_\ell v((k-l)\ell+j) \right] \leq 0.$$

Since $v(k\ell+j)$ is nondecreasing, we have from (10) and (11)

$$(12) \quad \Delta_\ell(z(k-1)\ell+j) + \gamma \frac{z(k\ell+j) \Delta_\ell v((k-l)\ell+j)}{v((k+1-l)\ell+j)} + b^*q(k\ell+j) \leq 0$$

for $\gamma > 0$ and $k \geq K+l+1$. Since $\Delta_\ell((a(k-1)\ell+j) \Delta_\ell(v(k-1)\ell+j)^k) \leq 0$, we have $(a(k-1)\ell+j) (\Delta_\ell(v(k-1)\ell+j))^\gamma \geq (a(k-1)\ell+j) (\Delta_\ell(v(k-1)\ell+j))^\gamma$ and so (12) gives for $k \geq N_1 \geq K+l+1$

$$(13) \quad \Delta_\ell(z(k-1)\ell+j) + \gamma \frac{z(k\ell+j) z^{1/\gamma}(k\ell+j)}{a^{1/\gamma}((k-l)\ell+j)} + b^*q(k\ell+j) \leq 0.$$

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Multiply (13) by $h(k\ell + j)$ and summing, we obtain

$$\begin{aligned}
& b^* \sum_{s=K_1}^k h(s\ell + j)q(s\ell + j) \\
& \leq - \sum_{s=K}^k h(s\ell + j)\Delta_\ell(z(s-1)\ell + j) - \gamma \sum_{s=K_1}^k \frac{h(s\ell + j)(z(s\ell + j))^{\frac{\gamma+1}{\gamma}}}{a^{1/\gamma}((s-1)\ell + j)} \\
& = -h((k+1)\ell + j)z(k\ell + j) + h(K_1\ell + j)z((K_1-1)\ell + j) \\
& \quad - \sum_{s=K_1}^k \frac{\gamma h(s\ell + j)}{a^{1/\gamma}((s-1)\ell + j)} \left[(z(s\ell + j))^{\frac{\gamma+1}{\gamma}} - \frac{\Delta_\ell h(s\ell + j)}{\gamma h(s\ell + j)} \right. \\
& \quad \times a^{1/\gamma}((s-1)\ell + j)z(s\ell + j) + \frac{1}{\gamma} \left(\frac{\Delta_\ell h(s\ell + j)}{(\gamma+1)h(s\ell + j)} \right)^{\gamma+1} a^{\frac{\gamma+1}{\gamma}}((s-1)\ell + j) \left. \right] \\
& \quad + \sum_{s=K_1}^k \frac{(\Delta_\ell h(s\ell + j))^{\gamma+1}}{(\gamma+1)^{\gamma+1}h^\gamma(s\ell + j)}(a(s-1)\ell + j).
\end{aligned}$$

Using Lemma 4 we obtain

$$\begin{aligned}
& b^* \sum_{s=K_1}^k h(s\ell + j)q(s\ell + j) \leq h(K_1\ell + j)z((K_1-1)\ell + j) \\
& \quad + \sum_{s=K_1}^k \frac{(a(s-l)\ell + j)(\Delta_\ell h(s\ell + j))^{\gamma+1}}{(\gamma+1)^{\gamma+1}h^\gamma(s\ell + j)}.
\end{aligned}$$

This gives contradiction to conditions (4) and (5) and the proof is complete. $\square$

Next assume that conditions (i) and (ii) are satisfied. Let $h_0(k\ell + j) = \sum_{i=k+1}^{\infty} q(i\ell + j)$ and $\left| \frac{f(x)}{x^\gamma} \right| \geq M > 0$ for $|x| \geq \delta > 0$. Define

$$h_r(k\ell + j) = M^{\frac{\gamma+1}{\gamma}} \gamma \sum_{i=k+1}^{\infty} \frac{(h(r-1(i)))^{1+1/\gamma}}{a(i-l)^{1/\gamma}} + Mh_0(k\ell + j), \quad r \in (0, \infty)$$

and

$$E(k\ell + j) = \sum_{i=1}^k \frac{1}{(a((i-l)\ell + j))^{1/\gamma}}.$$

Theorem 6. Assume conditions (i), (ii), (iii) and (iv) hold. Further assume that one of the following conditions hold.

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- (a) $\sum_{k=1}^{\infty} q(k\ell + j) = +\infty$, that is $h_0(0)$ does not exist;
 (b) there exists a positive integer j such that $h_n(k\ell + j)$ is defined for $n \in [0, r-1]$ but $h_r(O)$ does not exist.

Then every solution of equation (1) is oscillatory.

Proof. Assume that $v(k\ell + j)$ is a non oscillatory solution of equation (1) with $v(k\ell + j) > 0$ for $k > K + l$. By Lemma 3,

(a). $\Delta_{\ell} v(k\ell + j) > 0$ for $k \geq N + l$. Now define

$$W((k-1)\ell + j) = \frac{(a(k-1)\ell + j)(\Delta_{\ell}(v((k-1)\ell + j)))^{\gamma}}{v^{\gamma}((k-l)\ell + j)}$$

for $k \geq K + l$. Then from the proof of Theorem 5, we obtain

$$(14) \quad \begin{aligned} & \Delta_{\ell} W((k-1)\ell + j) + \gamma \frac{W^{1+\frac{1}{\gamma}}(k\ell + j)}{(a^{1/\gamma}(k-l)\ell + j)} \\ & + Mq(k\ell + j) \leq 0, \quad k \geq K_1 \geq K + l + 1. \end{aligned}$$

Summing the last inequality from K_1 to k , we obtain

$$(15) \quad \begin{aligned} & W(k\ell + j) + \gamma \sum_{s=K_1}^k \frac{W^{1+\frac{1}{\gamma}}(s\ell + j)}{(a^{1/\gamma}(s-l)\ell + j)} \\ & + M \sum_{s=K_1}^k q(s\ell + j) \leq W((K_1-1)\ell + j). \end{aligned}$$

If $\sum_{s=K_1}^{\infty} \frac{W^{1+\frac{1}{\gamma}}(s\ell + j)}{a^{1/\gamma}((s-l)\ell + j)} = +\infty$ then $\lim_{k \rightarrow \infty} \left(1 + \frac{W(k\ell + j)}{\gamma \sum_{s=K_1}^k \frac{W^{1+\frac{1}{\gamma}}(s\ell + j)}{(a^{1/\gamma}(s-l)\ell + j)}} \right) \leq 0$
 or $\lim_{k \rightarrow \infty} \sup \frac{W(k\ell + j)}{\gamma \sum_{s=K_1}^k \frac{W^{1+\frac{1}{\gamma}}(s\ell + j)}{a^{1/\gamma}((s-l)\ell + j)}} \leq -1$, which contradicts $W(k\ell + j) > 0$.

Hence

$$\sum_{s=K_1}^{\infty} \frac{W^{1+\frac{1}{\gamma}}(s\ell + j)}{a^{1/\gamma}((s-l)\ell + j)} < \infty.$$

On the other hand from (14) we have $\Delta_{\ell}(W(k-1)\ell + j) \leq 0$ which imply $\lim_{k \rightarrow \infty} W(k\ell + j) = c \geq 0$. If $c > 0$ then it would imply that

$\sum_{s=K_1}^{\infty} \frac{1}{a^{1/\gamma}((s-l)\ell + j)} < \infty$, a contradiction. Hence $\lim_{k \rightarrow \infty} W(k\ell + j) = 0$.

(a) If $h_0(0) = +\infty$, then from (15) we have $W(k\ell + j) \leq \alpha W((K_1 -$

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1) $\ell + j) - M \sum_{s=K_1}^k q(s\ell + j)$. This gives a contradiction.

(b) If $h_n(k\ell + j)$ is defined for $n \in [0, r - 1]$, letting $k \rightarrow \infty$ in (15), we have

$$(16) \quad W(k\ell + j) \geq \gamma \sum_{s=k+1}^{\infty} \frac{W^{1+\frac{1}{\gamma}}(s\ell + j)}{a^{1/\gamma}((s-l)\ell + j)} + Mh_0(k\ell + j).$$

and hence $W(k\ell + j) \geq Mh_0(k\ell + j)$. If $r = 1$ then

$$\sum_{s=k+1}^{\infty} \frac{h_0^{1+\frac{1}{\gamma}}(s\ell + j)}{a^{1/\gamma}((s-l)\ell + j)} \leq \frac{1}{M^{1+\frac{1}{\gamma}}} \sum_{s=k+1}^{\infty} \frac{W^{1+\frac{1}{\gamma}}(s\ell + j)}{a^{1/\gamma}((s-l)\ell + j)} < \infty$$

which contradicts $h_1(0) = +\infty$. If $r > 1$, then

$$\begin{aligned} h_1(k\ell + j) &= M^{1+\frac{1}{\gamma}} \gamma \sum_{i=k+1}^{\infty} \frac{h_0(i\ell + j)}{(a^{1/\gamma}(i-l)\ell + j)} + Mh_0(k\ell + j) \\ &\leq \gamma \sum_{i=k+1}^{\infty} \frac{W^{1+\frac{1}{\gamma}}(i\ell + j)}{(a^{1/\gamma}(i-l)\ell + j)} + Mh_0(k\ell + j) \leq W(k\ell + j). \end{aligned}$$

In a similar fashion, we can prove $W(k\ell + j) \geq h_k(k\ell + j)$ for $n \in [1, r - 1]$. Therefore

$$\sum_{i=k+1}^{\infty} \frac{h^{1+\frac{1}{\gamma}}((r-1)\ell + j)(i\ell + j)}{(a^{1/\gamma}(i-l)\ell + j)} \leq \sum_{i=k+1}^{\infty} \frac{W^{1+\frac{1}{\gamma}}(i\ell + j)}{(a^{1/\gamma}(i-l)\ell + j)} < \infty$$

which contradicts (b). $\square$

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