

Pair Sum Labelings of Zero Divisor Graphs

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August 9, 2017

Abstract

Let G be a (r, s) graph. An one to one map $f : V(G) \rightarrow \{\pm 1, \pm 2, \dots, \pm r\}$ is called a pair sum labeling if the induced edge mapping, $f_e : E(G) \rightarrow Z - \{0\}$ defined by $f_e(uv) = f(u) + f(v)$ is one-one and $f_e(E(G))$ is either of the form $\{\pm l_1, \pm l_2, \dots, \pm l_{s/2}\}$ or $\{\pm l_1, \pm l_2, \dots, \pm l_{(s-1)/2}\} \cup \{l_{(s+1)/2}\}$ according as S is even or odd. A graph with a pair sum labeling defined on it is called pair sum graph. In this paper, we evaluate the pair sum labelings of zero divisor graphs.

AMS Subject Classification: 05C25, 05C69

Key Words and Phrases: Labelings, Zero divisor graph.

1 Pair Sum Labeling of $\Gamma(Z_n)$

The first simplification of Beck's[3] zero divisor graph was introduced by D. F. Anderson and P. S. Livingston [2]. Their motivation was to give a better illustration of the zero divisor structure of the ring. In this new zero divisor graph, which is still a simple graph with edges defined the same way as above, only the zero divisors of the ring are included. That is, non-zero elements r of R such that $\text{Ann}_R(r) \neq 0$. Given a ring R , let $Z^*(R)$ denote the set of zero divisors of R . Let $Z^*(R)$ denote the graph whose vertex set is $Z^*(R)$, such that distinct vertices r and s are adjacent provided that $rs = 0$. Anderson and Livingston often focus on the case when R is finite, as these rings yield finite graphs. D. F. Anderson and P. S. Livingston, and others, e.g., [1, 2, 3, 5] investigate the interplay between the graph theoretic properties of $\Gamma(R)$ and the ring theoretic properties of R . Pair sum labeling satisfies the following observation [4].

- (i) If f is a pair sum labeling defined on $\Gamma(Z_n)$ then $\sum_{u \in V(G)} d(u)f(u) = 0$ iff G is of even size.
- (ii) If f is a pair sum labeling then x and $-x$ are not labels of two adjacent vertices. [Otherwise, Zero appears as an edge label].
- (iii) If $\Gamma(Z_n)$ is an even size pair sum graph then $\Gamma(Z_n) - e$ is also a pair sum graph for every edge e .
- (iv) Let $\Gamma(Z_n)$ be an odd size pair sum graph with $-f_e(e) \notin f_e(E)$. Then $\Gamma(Z_n) + e$ is a pair sum graph.
- (v) Let $\Gamma(Z_n)$ be a pair sum graph with even size and let f be a pair sum labeling of G with $f(n) = M$. Then the graph $\Gamma(Z_n)^*$ with $V(\Gamma(Z_n)^*) = V(\Gamma(Z_n)) \cup \{v\}$ and $E(\Gamma(Z_n)^*) = E(\Gamma(Z_n)) \cup \{uv\}$ is also a pair sum graph.
- (vi) Let $\Gamma(Z_n)$ be an even order and even size graph. If G is super vertex graceful then G is a pair sum graph.
- (vii) Every graph is a subgraph of a connected pair sum graph.

Theorem 1. *Every $\Gamma(Z_n)$ is a subgraph of a connected pair sum graph.*

Proof: Let $\Gamma(Z_n)$ be a graph with $(p - 1)$ vertices. Take two copies of the $\Gamma(Z_{p^2})$. Let u_i be the vertex of the first copy of $\Gamma(Z_{p^2})$ and v_i be the vertex of the second copy $\Gamma(Z_{p^2})$, where $1 \leq i \leq p - 1$.

$$m = 2^{p-1} - 2(p-1).$$

Let $V(\Gamma(Z_{2p})) = \{w, w_i; 1 \leq i \leq p-1\}$ and $E(\Gamma(Z_{2p})) = \{w, w_i; 1 \leq i \leq p-1\}$.

The graph $\Gamma(Z_n)^*$ is obtained from $\Gamma(Z_{p^2}) \cup \Gamma(Z_{p^2}) \cup \Gamma(Z_{2p})$ by joining the vertex w to v_1 and v_2 .

Obviously $\Gamma(Z_n)$ is a subgraph of $\Gamma(Z_n)^*$. Assign the label 2^{i+1} to $u_i (1 \leq i \leq p-1)$ and -2^{i+1} to $v_i (1 \leq i \leq p-1)$. Label the vertices $w_1, w_2, \dots, w_{(p-1)/2}$ by $1, 3, 5, \dots, p-2$ and $w_{((p-1)/2)+1}, w_{((p-1)/2)+2}, \dots, w_{p-1}$ by $-5, -7, \dots, -(p+2)$. Finally assign the integer 2 to w . Clearly, thus labeling is a pair sum labeling of $\Gamma(Z_n)^*$.

Theorem 2. *If $\Gamma(Z_n)$ is a pair sum graph then $\Gamma(Z_n) \cup m\Gamma(Z_4)$ is also a pair sum graph.*

Proof: Let f be a pair sum labeling of $\Gamma(Z_n)$. Let $V(\Gamma(Z_n)) = \{u_i : 1 \leq i \leq k\}$ and $V(\Gamma(Z_n)) = \{v_i : 1 \leq i \leq m\}$. Define $g : V(\Gamma(Z_n) \cup m\Gamma(Z_4)) \rightarrow \{\pm 1, \pm 2, \dots, \pm(k+m)\}$ by $g(u_i) = f(u_i)$ and $g(v_i) = k+i (1 \leq i \leq m)$. Then clearly, g is a pair sum labeling of $\Gamma(Z_n) \cup m\Gamma(Z_4)$.

Theorem 3. *If $\Gamma(Z_n)$ is a (r, s) pair sum graph then $s \leq 4r-2$.*

Proof: Let f be a pair sum labeling of $\Gamma(Z_n)$. Obviously $-2r+1 \leq f_e(uv) \leq 2r-1$, and $f_e(uv) \neq 0$ for all uv . Hence, we proved $s \leq 4r-2$.

Theorem 4. *For any path in $\Gamma(Z_n)$ is a pair sum graph.*

Proof: We know that, the only path in $\Gamma(Z_n)$ are $\Gamma(Z_4), \Gamma(Z_8)$ and $\Gamma(Z_9)$. If $n=8$, then vertex set of $\Gamma(Z_n)$ is $\{2, 4, 6\}$, then the vertex 4 is adjacent to 2 and 6. That is $2.4=0$ and $4.6=0$. Thus $\Gamma(Z_8)$ is isomorphic to P_3 .

Similarly, if $n=9$, then the vertex set of $\Gamma(Z_n)$ is $\{3, 6\}$ and $\Gamma(Z_9)$ is isomorphic to P_2 . Since, we know, $\Gamma(Z_4)$ is isomorphic to P_1 .

$\Gamma(Z_4) \simeq P_1$. Assign any value.

$\Gamma(Z_9) \simeq P_2$. $\Gamma(Z_8) \simeq P_3$.

2 Pair sum labeling of complete and complete bipartite graphs in $\Gamma(Z_n)$

Theorem 5. *The complete graph of $\Gamma(Z_{p^2})$ is a pair sum graph if and only if $p \leq 5$, where p is any prime number.*

Proof: Using the previous theorem, we know that k_1, k_2 are pair sum graphs. Because k_1 is isomorphic to $\Gamma(Z_4)$ and k_2 is isomorphic to $\Gamma(Z_9)$. In $\Gamma(Z_n)$, there is no graph isomorphic to k_3 . That is, no $\Gamma(Z_n)$, has a unique cycle with any length.

The proof is by the method of indirect, Let us assume that $p > 5$. Suppose f is a pair sum labeling. Then, if f is a pair sum labeling then x and $-x$ are not labels of two adjacent vertices. Clearly either x and $-x$ must be a vertex label for all $x \leq p$.

Case(i): 1 is a vertex label.

Sub case(a): $(p-1)$ is a vertex label.

Claim(i): $p-2$ is not a vertex label.

Suppose $(p-2)$ is a vertex label. Then $2p-3$ is an edge label. But $-2p+3$ is not an edge label. By the observation, if f is a pair sum labeling then x and $-x$ are not labels of two adjacent vertices, then $-p+2$ is a vertex label.

Claim(ii): $p-3$ is not a vertex label.

Suppose $(p-3)$ is a vertex label. Then $2p-4$ is an edge label. To get the edge label $-2p+4$, both $-p+1$ and $-p+3$ must be vertex label. This is not possible. Therefore, $-p+4$ is a vertex label.

Claim(iii): $p-4$ is not a vertex label.

Suppose $(p-4)$ is a vertex label. If 2 were a vertex label then $-p+4$ occurs as an edge label twice. If -2 were a vertex label then $p-3$ occurs an edge label which is not possible by Claim(ii).

Claim(iv): $-p+4$ is not a vertex label.

In case $-p+4$ is a vertex label, 3 or 1 or $-p+1$ occurs as an edge label at least twice. Claims (i) and (iv) contradict each other.

Sub case(b): $(-p+1)$ is a vertex label.

Similar arguments as in sub case(a), $(-p+2), (-p+3)$ are not vertex labels.

Claim(v): $-p+4$ is not a vertex label.

If $-p+4$ is a vertex label, then $-2p+5$ is an edge label. But $2p-5$ is not an edge label, therefore $-p+4$ is not a vertex label.

Claim(vi): $p-4$ is not a vertex label.

Suppose, $p - 4$ is a vertex label. If 2 is a vertex label then an edge label $(p - 1)$ occurs twice. If -2 is a vertex label, the edge label -1 occurs twice. By claim 5 6, we get a contradiction. **Case(ii):** -1 is a vertex label.

Subcase(c): $(p - 1)$ is a vertex label.

Claim(vii): $-p + 2$ is not a vertex label.

If $-p + 2$ is a vertex label then $-2p + 3$ is an edge label. But $2p - 3$ is not an edge label, therefore $-p + 2$ is not a vertex label. This implies $p - 2$ is a vertex label.

Claim(viii): $-p + 3$ is not a vertex label.

Suppose $-p + 2$ is a vertex label, then $-2p + 4$ is an edge label. But $2p - 4$ is not an edge label, therefore $-2p + 4$ is not a vertex label. This implies that $p - 3$ is a vertex label.

Claim(ix): $p - 4$ is not a vertex label.

If 2 and 3 are vertex label $(p - 1)$ occurs twice as an edge label. If 2 and -3 are vertex label then $p - 5$ occurs twice as an edge label. If -2 is a vertex label, then -3 occurs twice as an edge label, therefore $p - 4$ is not an edge label.

Claim(x): $-p + 4$ is not a vertex label.

Suppose $-p + 4$ is a vertex label. If 2 is a vertex label then 1 occurs twice as an edge label. If -2 is a vertex label then $p - 4$ occurs twice as an edge label. By claims (ix) and (x), we get a contradiction.

Subcase(d): $-p + 1$ is a vertex label.

Using previous Sub cases, we know that $p - 2$ and $p - 3$ are not vertex labels.

Claim(xi): $p - 4$ is not a vertex label.

Suppose $p - 2$ is a vertex label. If 2 is a vertex label then $p - 2$ occurs twice as an edge label. If -2 is a vertex label then $-p + 1$ occurs as an edge label twice. Therefore $p - 4$ is not an edge label.

Claim(xii): $-p + 4$ is not a vertex label.

Suppose $-p + 4$ is a vertex label. If 2 is a vertex label then 1 occurs twice as an edge label. If -2 is a vertex label then $-p + 1$ occurs twice as an edge label. By claims (xi) and (xii), we get a contradiction. By cases (i) and (ii), neither 1 nor -1 is not a vertex label. Therefore $\Gamma(Z_{p^2}), p > 5$ is not a pair sum graph.

Theorem 6. For any prime number $p > 2$, $\Gamma(Z_{2p})$ is a pair sum graph.

Proof: The vertex set of $\Gamma(Z_{2p})$ is $\{2, 4, 6, \dots, 2(p-1), p\}$ and the edge set of $\Gamma(Z_{2p})$ is $\{2p, 4p, \dots, 2(p-1)p\}$. Let f be a function from vertex set of $\Gamma(Z_{2p})$ to the labeling set $\{\pm 1, \pm 2, \dots, \pm p\}$. We know that $\Gamma(Z_{2p})$ is isomorphic to $K_{1,p-1}$. Once again we define the vertex set of $K_{1,p-1}$ is $\{u, u_i : 1 \leq i \leq p-1\}$ and $E(K_{1,p-1}) = \{uu_i : 1 \leq i \leq p-1\}$.

Clearly, $f(u) = 1$

$$f(u_i) = i + 1, 1 \leq i \leq \lfloor \frac{p-1}{2} \rfloor$$

$$f(u_{\lfloor \frac{p-1}{2} \rfloor + i}) = -i - 3, 1 \leq i \leq \lfloor \frac{p-1}{2} \rfloor$$

$$\text{Hence, } f_e(E(\Gamma(Z_{2p}))) = \{\pm 3, \pm 4, \dots, \pm \lfloor \frac{p+3}{2} \rfloor\}$$

Theorem 7. $\Gamma(Z_{pq})$ is a pair sum graph, where p and q are prime numbers with $p = 3$ and $q > p$.

Proof: The Vertex Set of $\Gamma(Z_{pq})$ is $\{p, 2p, \dots, p(q-1), q, 2q, \dots, q(p-1)\}$ and the edge set of $\Gamma(Z_{pq})$ is $\{pq, p-2q, \dots, p(q-1).q(p-1)\}$. Clearly, we know that $\Gamma(Z_{pq})$ is isomorphic to $K_{p-1,q-1}$. Let $p = 3$ the $\Gamma(Z_{pq})$ is $K_{2,q-1}$ graph.

Let $V(\Gamma(Z_{pq})) = \{u, v, w_i : 1 \leq i \leq q-1\}$ and $E(\Gamma(Z_{pq})) = \{uw_i, vw_i : 1 \leq i \leq q-1\}$.

Define $f : V(\Gamma(Z_{pq})) \rightarrow \{\pm 1, \pm 2, \dots, \pm(q+1)\}$ by

$$f(u) = -1$$

$$f(v) = -(q+1)$$

$$f(w_i) = i + 1; 1 \leq i \leq q-1$$

Thus, $f_e(E(\Gamma(Z_{pq}))) = \{\pm 1, \pm 2, \dots, \pm(q-1)\}$ and hence $\Gamma(Z_{pq})$ is a pair sum graph, where $p = 3$ and $q > p$ is any prime.

Theorem 8. $\overline{\Gamma(Z_{p^2})} + 2\Gamma(Z_9)$ is a pair sum graph, where p is any prime number.

Proof: Let $\{v_1, v_2, \dots, v_{p-1}\}$ be the vertices of $\Gamma(Z_{p^2})$ and u, v, w, z be the vertices in $2\Gamma(Z_9)$. Let $V(\overline{\Gamma(Z_{p^2})} + 2\Gamma(Z_9)) = V(\overline{\Gamma(Z_{p^2})}) \cup V(2\Gamma(Z_9))$ and $E(\overline{\Gamma(Z_{p^2})} + 2\Gamma(Z_9)) = \{uv, wz, uv_i, vv_i, ww_i, zv_i : 1 \leq i \leq (p-1)\}$.

Define $f : V(\overline{\Gamma(Z_{p^2})} + 2\Gamma(Z_9)) \rightarrow \{\pm 1, \pm 2, \dots, \pm(p+3)\}$ by

$$f(v_i) = 2i - 1, 1 \leq i \leq \frac{p-1}{2}$$

$$f(v_{\frac{p-1}{2} + i}) = -(2i - 1), 1 \leq i \leq \frac{p-1}{2}$$

$$f(u) = p - 1$$

$$f(v) = p + 2$$

$$f(w) = -p + 1$$

$$f(z) = -p - 2$$

Hence, $f_e(E(\overline{\Gamma(Z_{p^2})} + 2\Gamma(Z_9)))$

$$= \{p, p+2, p+4, \dots, 2p-2\} \cup \{-p, -p-2, -p-4, \dots, -2p+2\} \cup \\ \{p-2, p-4, p-6, \dots, 1\} \cup \{-p+2, -p+4, \dots, -1\} \cup \{p+3, p+ \\ 7, \dots, 2p\} \cup \{-p-3, -p-7, \dots, -2p\} \cup \{p+1, p-1, p-3, \dots, 2\} \cup \\ \{-p-1, -p+1, -p+3, \dots, -2\} \cup \{2p+1, -2p-1\}.$$

Therefore, f is a pair sum labeling of $\overline{\Gamma(Z_{p^2})} + 2\Gamma(Z_9)$.

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