

EVEN VERTEX MAGIC TOTAL LABELING

C.T. NAGARAJ

Research Scholar, Department of Mathematics

Research and Development centre

Bharathiar university, Coimbatore-641046, Tamilnadu, India.

e – mail : ct_nagaraaj@yahoo.co.in

C.Y. PONNAPPAN

Department of Mathematics

Government Arts College, Paramakudi - 623707, Tamilnadu, India.

e – mail : pons_mdu1969@yahoo.com

G. PRABAKARAN

Department of Mathematics

Thiagarajar College, Madurai - 625009, Tamilnadu, India.

e – mail : tcmaths@yahoo.in

ABSTRACT. A vertex-magic total labeling is an assignment of the integers from $1, 2, 3, \dots, m + n$ to the vertices and edges of G so that at each vertex, the vertex label and the labels on the edges incident at that vertex add to a fixed constant. In this paper, we introduce a new concept even vertex total labeling of a graph. We study some of the basic properties of such labelings, find some families of graphs that admit an even vertex-magic total labelings and show that some other families of graphs do not.

Key words: Evenvertex magic labeling, even vertex magic graph.

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1. INTRODUCTION

In this paper, we consider only finite simple undirected graphs. The graph G has vertex set $V = V(G)$ and edge set $E = E(G)$ and we take $m = |E|$ and $n = |V|$. The set of vertices adjacent to a vertex u of G is denoted by $N(u)$.

A labeling of a graph G is a mapping that carries a set of graph elements, usually the vertices and edges into a set of numbers, usually integers. Many kinds of labeling have been studied and an excellent survey of graph labeling can be found

in [3].

Sedlacek [4] introduced the concept of magic labeling. Suppose that G is a graph with m edges. We will say that G is magic if the edges of G can be labeled by the numbers $1, 2, 3, \dots, m$ so that the sum of labels of all the edges incident with any vertex is the same.

MacDougall [2] introduced the concept of vertex magic total labeling. If G is a finite simple undirected graph with n vertices and m edges, then a vertex magic total labeling is a bijection f from $V(G) \cup E(G)$ to the integers $1, 2, \dots, m+n$ with the property that for every u in $V(G)$, $f(u) + \sum_{v \in N(u)} f(uv) = k$ for some constant k . They studied the basic properties of vertex magic graphs and showed some families of graphs having a vertex magic total labeling.

MacDougall [2] also introduced the concept of super vertex magic total labeling. They call a vertex magic total labeling is super if $f(V(G)) = 1, 2, 3, \dots, n$. In this labeling, the smallest labels are assigned to the vertices.

Swaminathan and Jeyanthi [6] introduced a concept with same name of super vertex magic labeling. They call a vertex magic total labeling is super if $f(E(G)) = 1, 2, 3, \dots, m$. Note that the smallest labels are assigned to the edges. But Marimuthu and Balakrishnan [7] called the above type of labeling as E-super vertex magic total labeling. They proved the following: P_n is super vertex magic if and only if n is odd and $n \geq 3$; C_n is super vertex magic if and only if n is odd; the star graph is super vertex magic if and only if it is S_2 ; and rC_s is super vertex magic if and only if r and s are odd.

The following definition and results that will subsequently be very useful to prove some theorems.

Definition 1.1. A sun S_n is a cycle C_n with an edge terminating in a vertex of degree 1 attached to each vertex. The sun S_n consists of vertex set $V(S_n) = \{v_i/1 \leq i \leq n \cup a_i/1 \leq i \leq n\}$ and edges set $E(S_n) = \{v_i v_{i+1}/1 \leq i \leq n \cup v_i a_i/1 \leq i \leq n\}$. We note that if $i = n$ then $i + 1 = 1$.

Definition 1.2. An (n, t) kite consists of a cycle of length n with a t -edge path (the tail) attached to one vertex.

Definition 1.3. A fan graph F_n is obtained from a path P_n by adding a new vertex and joining it to all the vertices of the path by an edge, the new edges are called the spokes of the fan.

Lemma 1.4. If G has a vertex magic total labeling then $m \geq \frac{2n}{3}$.

2. DEFINITION AND MAIN RESULTS

Definition 2.1. A vertex magic total labeling f is called an even vertex magic total labeling if

$$f : V \rightarrow \{2, 4, 6, 8, \dots, 2n\}$$

A graph G is called an even vertex magic if there exists an even vertex magic total labeling of G .

Example 2.2.

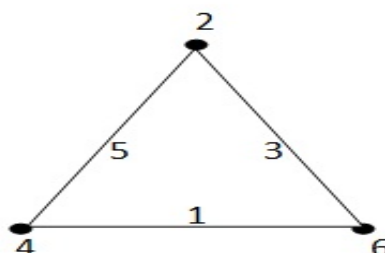
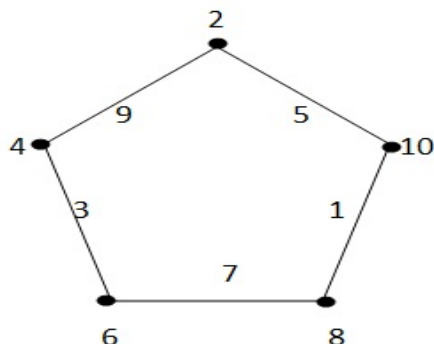


Figure 1: $n=3$, $m=3$, $k=10$

Figure 2: $n=5, m=5, k=16$

Theorem 2.3. *Let G be a nontrivial graph .If G is an even vertex magic, then the magic constant k is given by*

$$k = \frac{m^2 + 2mn + m}{n}$$

Proof. Let G be a nontrivial graph .

Let f be an even vertex magic total labeling of a graph G with the magic number k .

Then $f(V) = \{2, 4, 6, \dots, 2n\}$.

$$\begin{aligned} k &= f(u) + \sum_{v \in N(u)} f(uv) \forall u \in V \\ nk &= n(n+1)(m+n)(m+n+1) - 2n(n+1) \\ k &= \frac{m^2 + 2mn + m}{n}. \end{aligned} \tag{1}$$

□

Theorem 2.4. *If a graph G has an even vertex magic total labeling with constant k , then $k \geq \frac{16n+6}{9}$.*

Proof. This result directly follows from Lemma 1.4 and Theorem 2.3. □

Theorem 2.5. *A cycle C_n is an even vertex magic iff n is odd.*

Proof. Let $V(C_n) = \{v_1, v_2, v_3, \dots, v_n\}$ and $E(C_n) = \{e_i = v_i v_{i+1} / 1 \leq i \leq n-1\} \cup \{e_n = v_n v_1\}$.

Here $n = m$. Suppose C_n is an even vertex magic. Then by Theorem 2.3,

$$\begin{aligned} k &= \frac{m^2 + 2mn + m}{n} \\ k &= \frac{n^2 + 2n^2 + n}{n} \\ k &= 3n + 1. \end{aligned} \tag{2}$$

Suppose n is even. Then $k = 3n + 1$ is odd.

For any even vertex magic total labeling $f, k = f(u) + \sum_{v \in N(u)} f(uv) \forall u \in V$.

In particular, for $2 \leq i \leq n-1$,

$$f(v_i) + f(v_i v_{i-1}) + f(v_i v_{i+1}) = k$$

which is a contradiction, since $f(v_i)$ is an even number and $f(v_i v_{i-1})$ and $f(v_i v_{i+1})$ are odd numbers.

Therefore n is odd.

Conversely,

Suppose n is an odd number.

Let $V(C_n) = \{v_1, v_2, v_3, \dots, v_n\}$ and

$$E(C_n) = \{e_i = v_i v_{i+1} / 1 \leq i \leq n-1\} \cup \{e_n = v_n v_1\}$$

.

Define $f : V \cup E \rightarrow \{1, 2, 3, \dots, 2n\}$ as follows $f(v_i) = 2i, 1 \leq i \leq n$ and

$$f(e_i) = \begin{cases} 2n - i & \text{if } i \text{ is odd} \\ n - i & \text{if } i \text{ is even} \end{cases} \tag{3}$$

It is easily seen that f is an even vertex magic labeling with the magic number $3n + 1$.

□

3. EVEN VERTEX MAGIC TOTAL LABELLING ON A DISCONNECTED GRAPH

In this section we give an even vertex magic total labeling for the disconnected graph mC_n that is, the disjoint union of m cycles of length n , where m and n are odd.

Theorem 3.1. rC_s is an even vertex magic iff r and s are odd.

Proof. Assume that rC_s is even vertex magic.

Number of vertices of rC_s is rs .

Number of edges of rC_s is rs .

Then by Theorem 2.3,

$$k = 3rs + 1. \quad (4)$$

For any even vertex magic total labeling $f, k = f(u) + \sum_{v \in N(u)} f(uv) \forall u \in V$. Since any vertex of rC_s is adjacent to only two edges, $f(u) + f(e_1) + f(e_2) = k$, where e_1 and e_2 are the edges adjacent to u .

Here $f(u)$ is an even number and $f(e_1)$ and $f(e_2)$ are odd numbers.

Therefore k is even.

Therefore $k = 3rs + 1$ is even iff r and s are odd.

Conversely,

Assume that rc_s is even vertex magic.

By theorem, r and s both odd.

Let r and s be odd integers.

Assume that the graph rc_s has the vertex set $V = V_1 \cup V_2 \cup V_3 \cup \dots \cup V_r$, where $V_i = \{V_i^1, V_i^2, V_i^3, \dots, V_i^s\}$ and the edge set $E = E_1 \cup E_2 \cup E_3 \cup \dots \cup E_r$ where $E_i = \{e_i^1, e_i^2, e_i^3, \dots, e_i^s\}$, and $e_i^j = v_i^j v_i^{(j+1)}$ for $1 \leq i \leq r, 1 \leq j \leq s-1, e_i^s = v_i^s v_i^1$.

Define $f : V \cup E \rightarrow \{1, 2, 3, \dots, 2rs\}$ as follows:

$$f(v_i^1) = 2i, \quad i = 1, 2, \dots, r$$

$$F(v_i^2) = \begin{cases} 4r - 4i + 2, & 1 \leq i \leq \frac{(r-1)}{2} \\ 6r - 4i + 2, & \frac{(r+1)}{2} \leq i \leq r. \end{cases}$$

For $j = 3, 5, 6, \dots, s$

$$f(v_i^j) = \begin{cases} (2j-1)r + 2i + 1, & 1 \leq i \leq \frac{(r-1)}{2} \\ (2j-3)r + 2i + 1, & \frac{(r+1)}{2} \leq i \leq r. \end{cases}$$

$$f(e_i^1) = \begin{cases} 2rs + 2i - r, & 1 \leq i \leq \frac{(r-1)}{2} \\ 2rs - 3r + 2i, & \frac{(r+1)}{2} \leq i \leq r. \end{cases}$$

$j = 2, 4, 6, \dots, s-1$

$$f(e_i^s) = rs + 2i - 1 - (j+1)r \quad 1 \leq i \leq r.$$

$j = 3, 5, 7, \dots, s$

$$f(e_i^j) = \begin{cases} 2rs - 4i + 1 - (j-1)r, & 1 \leq i \leq \frac{(r-1)}{2} \\ 2rs - 4i + 1 - (j-3)r, & \frac{(r+1)}{2} \leq i \leq r. \end{cases}$$

It is easily verified that f is even vertex magic labeling of rC_s with magic constant $k = 3rs + 1$.

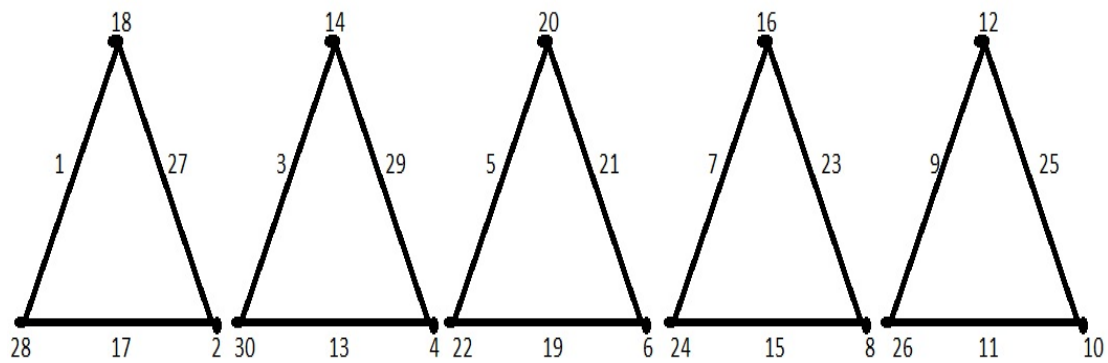


Figure 3: $n=15, m=15, k=46$

□

Theorem 3.2. For $n \geq 2$, the fan graph F_n is an even vertex magic if and only if $n = 2$.

Proof. Suppose there exists an even vertex magic labeling of F_n with magic constant k . Then by theorem 2.3, $k = \frac{8n^2-2}{n+1}$ which is not an integer for all n except 1,2,5.

Even vertex magic labeling for F_2 is

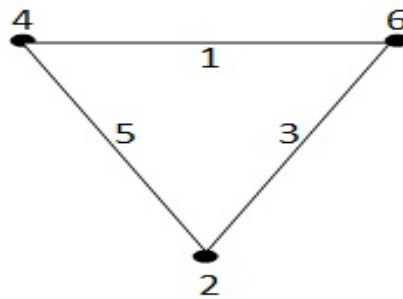


Figure 4: $n=3, m=3, k=10$

For F_5

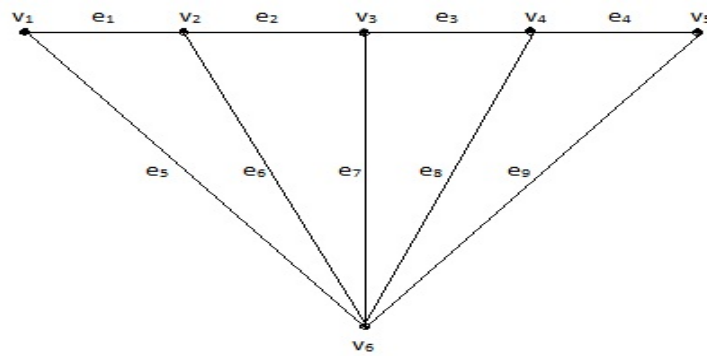


Figure 5

Suppose there exists an even vertex magic labeling for F_5 , with magic constant $k = 33$.

The fan graph F_5 with 6 vertices and 9 edges, $f(V) = \{2, 4, 6, 8, 10, 12\}$ and $f(E) = \{1, 3, 5, 7, 9, 11, 13, 14, 15\}$.

Here in the edge labels, 14 is the only even number.

If $f(e_i) = 14$, where $i \in \{1, 2, 3\}$, then $k = f(v_{i+1}) + f(e_i) + f(e_{i+1}) + f(e_{i+5})$, is an even number which is contradiction to $k = 33$.

If $f(e_4) = 14$, then $k = f(v_4) + f(e_3) + f(e_4) + f(e_8)$, is an even number which is a contradiction to $k = 33$.

If $f(e_i) = 14$, where $i \in \{5, 6, 7, 8, 9\}$, $k = f(v_6) + \sum_{i=5}^9 f(e_i)$, is an even number which is a contradiction to $k = 33$.

Therefore F_5 is not an even vertex magic.

□

4. EVEN VERTEX MAGIC TOTAL LABELING OF SUN S_p

Theorem 4.1. *For $p \geq 3$, Sun S_p has an even vertex magic.*

Proof. The Sun S_p has $2p$ vertices and $2p$ edges.

Then by Theorem 2.3

$$k = 6p + 1. \quad (5)$$

$f : V \rightarrow \{2, 4, 6, \dots, 4p\}$ defined as follows

$$f(v_i) = 2i, \quad i = 1, 2, 3, \dots, p$$

$$f(a_i) = \begin{cases} 2p + 2, & \text{for } i=1 \\ 4p - 2i + 4 & \text{for } i = 2, 3, \dots, p \end{cases}$$

Define $f : E \rightarrow \{1, 3, 5, \dots, 4p - 1\}$ as follows

$$f(v_i v_{i+1}) = \begin{cases} 2p - 2i + 1, & \text{for } i=1, 2, \dots, p-1 \\ 1 & \text{for } i=p \end{cases}$$

$$f(v_i a_i) = \begin{cases} 4p - 1, & \text{for } i=1 \\ 2p + 2i - 3 & \text{for } i=2, 3, \dots, p \end{cases}$$

It is easily seen that f is an even magic labeling with the magic number $k = 6p + 1$.

□

5. GRAPHS WITH NO EVEN VERTEX MAGIC TOTAL LABELING

Theorem 5.1. *All wheels $W_n, n \geq 3$ are not an even vertex magic.*

Proof. Suppose there exists an even vertex magic labeling for W_n with magic constant k . Then by Theorem 2.3, $k = \frac{8n^2+6n}{n+1}$ which is not an integer for all n except 1. □

Theorem 5.2. *(n, t) kite is not even vertex magic.*

Proof. Suppose that kite (n, t) is even vertex magic.

Let f be an even vertex magic labeling.

The number of vertices of kite (n, t) is $n + t$

The number of edges of kite (n, t) is $n + t$

Then by Theorem 2.3 ,

$$\begin{aligned} k &= \frac{m^2 + 2mn + m}{n} \\ k &= \frac{(n+t)^2 + 2(n+t)^2 + (n+t)}{n+t} \\ k &= 3n + 3t + 1. \end{aligned} \tag{6}$$

Let $v_0, v_1, v_2, \dots, v_{n-1}, v_0$ be a vertex sequence of C_n , a vertex u_1 is adjacent to v_0 and a vertex u_2 is adjacent to u_1 and so on, a vertex u_t is adjacent to u_{t-1} .

Case (i): n is odd and t is odd.

Therefore $k = 3n + 3t + 1$ is odd.

For any even vertex magic labeling $f, k = f(u) + \sum_{v \in N(u)} f(uv) \forall u \in V$.

Let v_i be any vertex $i = 1, 2, 3, \dots, n-1, f(v_i) + f(v_i v_{i-1}) + f(v_i v_{i+1}) = k$.

Since $f(v_i)$ is even and $f(v_i v_{i-1})$ and $f(v_i v_{i+1})$ are odd.

Therefore k is even , which is a contradiction to k is odd.

Case (ii): n is even and t is even.

Therefore $k = 3n + 3t + 1$ is odd.

Similar to Case (i), again we get a contradiction.

Case (iii): n is odd and t is even.

Therefore $k = 3n + 3t + 1$ is even.

For any even vertex magic labeling $f, k = f(u) + \sum_{v \in N(u)} f(uv) \forall u \in V$.

Let u_t be pendent vertex. Then $f(u_t) + f(u_{t-1}u_t) = k$.

Since $f(u_t)$ is even and $f(u_{t-1}u_t)$ is odd, k is odd, which is a contradiction to k is even.

Case (iv): n is even and t is odd.

Therefore $k = 3n + 3t + 1$ is even.

Similar to Case (iii), again we get a contradiction.

In all the cases, we get a contradiction.

Therefore, any kite (n, t) is not an even vertex magic.

□

Theorem 5.3. *If $p \equiv 0(mod 4)$, then the complete graph K_p is not an even vertex magic.*

Proof. The complete graph K_p has p vertices and $\frac{p(p-1)}{2}$ edges.

Assume that $p \equiv 0(mod 4)$ and $p \geq 3$.

Then by Theorem 2.3

$$\begin{aligned} k &= \frac{p(p-1)^2}{4} + \frac{(p-1)}{2} + p(p-1) \\ k &= \frac{(p-1)(p+1)(p+2)}{4}. \end{aligned} \tag{7}$$

If $p \equiv 0(mod 4)$ then k is not an integer, which is a contradiction.

□

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REFERENCES

- [1] Alison M, Marr and W.D. Wallis, *Magic Graphs*, Birkhauser, 2013.
- [2] J.A. MacDougall, M. Miller, K.A. Sugeng, *Super vertex magic total labelling of graphs*, in: Proc. of the 15th Australian Workshop on Combinatorial Algorithms, 2004, pp. 222-229.
- [3] J.A. Gallian, *A dynamic survey of graph labeling*, Electron J. Combin. 16 (2009) DS6.
- [4] J. Sedlacek, Problem 27, in: *Theory of Graphs and its Applications*, Proc. Symposium, 1963, pp 163-167.
- [5] J.A. MacDougall, M. Miller, Slamin, W.D. Wallis, *Vertex magic total labeling of graphs*, Util. Math 61 (2002) 3-21.
- [6] V. Swaminathan, P. Jeyanthi, *Super vertex magic labeling*, Indian J. Pure Appl. Math 34(6)(2003) 935-939.
- [7] G. Marimuthu, M. Balakrishnan, *E-Super vertex magic labelings of graphs*, *Discrete Applied Mathematics*, 160, 1766-1744, 2012.

