ABSTRACT: Let G be a simple graph with n vertices and m edges and \( G^c \) be its complement. Let \( \delta(G) = \delta \) and \( \Delta(G) = \Delta \) be the minimum degree and the maximum degree of vertices of G, respectively. In this paper we present a sharp upper bound for the Laplacian spectral radius of complete bipartite graph \( K_{r,s} \) for \( s \geq r \) as \( \lambda_1(G) \leq \frac{s + \sqrt{s^2 + 8rs}}{2} \) and for \( r = s \), \( \lambda_1(G) = 2s \). We prove the upper bound of Nordhaus - Gaddum type relation on the Laplacian spectral radius of this graph as \( \lambda_1(G) + \lambda_1(G^c) \leq \frac{s + \sqrt{s^2 + 8rs}}{2} + (s - 1) \leq (3s - 1) \).

KEY WORDS: Laplacian spectral radius, Complete bipartite graph, Nordhaus - Gaddum.

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1 INTRODUCTION

Let \( G = (V, E) \) be a simple undirected graph with \( n \) vertices and \( m \) edges. The complement \( G^c \) of the graph \( G \) is the graph with the same vertex set as \( G \), where any two distinct vertices are adjacent
if and only if they are non adjacent in G. For \( v \in V \), the degree of \( v \), written by \( d(v) \), is the number of edges incident with \( v \). Let \( \delta(G) = \delta \) and \( \Delta(G) = \Delta \) be the minimum degree and the maximum degree of vertices of G, respectively. Moreover, we will use the symbols \( u \ adj v \) and \( u \ nadj v \) to denote \( u \) and \( v \) are adjacent or not adjacent, respectively. Let \( A(G) \) be the adjacency matrix of \( G \) and \( D(G) = \text{diag} \left( d(v_1), d(v_2), ..., d(v_n) \right) \) be the diagonal matrix of vertex degrees. The Laplacian matrix of \( G \) is \( L(G) = D(G) - A(G) \). Clearly, \( L(G) \) is a real symmetric matrix. From this fact and Gergorins Theorem, it follows that its eigenvalues are nonnegative real numbers. The largest eigenvalue of Laplacian matrix of graph \( G \) denoted as \( \lambda_1(G) \) is the Laplacian spectral radius of \( G \). Let \( G \) be a graph with the degree diagonal matrix \( D(G) \) and adjacency matrix \( A(G) \). The signless Laplacian matrix of \( G \) is denoted as \( Q(G) = D(G) + A(G) \). The largest eigenvalue of signless Laplacian matrix of \( G \) denoted as \( \lambda_1(Q) \) is the signless Laplacian spectral radius of \( G \).

Some upper bounds for \( \lambda_1(G) \) were given by Anderson and Morley [2], Li and Zhang [6, 8], Merris [11] and Rojo et al[15]. Shi [10] applies Nordhaus - Gaddum type relations for some important classes of graphs and derives the sharp bounds.

### 1.1 PRELIMINARIES

Before stating and proving our results, we present basic definitions and preliminary results which will be used for the subsequent discussions.

**Definition 1.** A graph \( G = (V,E) \) in which \( V \) can be partitioned into two subsets \( V_1 \) and \( V_2 \) so that each edge in \( G \) connects some vertex in \( V_1 \) to some vertex in \( V_2 \) is a bipartite graph. A bipartite simple graph is called complete if each vertex in \( V_1 \) is connected to each vertex in \( V_2 \). If \( |V_1| = r \) and \( |V_2| = s \), the corresponding complete bipartite graph is represented as \( K_{r,s} \).

**Example** We shall illustrate the complete bipartite graph \( K_{3,4} \) and its complement which comprises of disjoint union of \( K_3 \) and \( K_4 \).
Figure 1: Complete bipartite graph $K_{3,4}$ and its complement $K_3$ and $K_4$.

**Theorem 2.** [4] Let $G$ be a graph. Then

$$
\lambda_1(G) \leq \lambda_1(Q)
$$

Moreover, if $G$ is connected, then the equality holds if and only if $G$ is a bipartite graph.

**Theorem 3.** [4] Let $B$ be a real symmetric $n \times n$ matrix, and let $\lambda$ be an eigenvalue of $B$ with an eigenvector $x$ all of whose entries are nonnegative. Denote the $i$th row sum of $B$ by $s_i(B)$. Then

$$
\min_{1 \leq i \leq n} s_i(B) \leq \lambda \leq \max_{1 \leq i \leq n} s_i(B).
$$

**Theorem 4.** [4] Let $G$ be an $n$-vertex graph, $Q = Q(G)$ and $P$ any polynomial. Then

$$
\min_{v \in V(G)} s_v(P(Q)) \leq P(\lambda_1(Q)) \leq \max_{v \in V(G)} s_v(P(Q))
$$

**Theorem 5.** [4]. Let $G$ be a simple graph with $n$ vertices and $m$ edges. Let $\delta$ and $\Delta$ be the minimum degree and the maximum degree of $G$, respectively, and $\lambda_1(G)$ be the Laplacian spectral radius of $G$. Then

$$
\lambda_1(G) \leq \frac{(\Delta + \delta - 1)\sqrt{(\Delta + \delta - 1)^2 + 4(4m - 2\delta(n - 1))}}{2}
$$

Moreover, if $G$ is connected, then the equality holds if and only if $G$ is a regular bipartite graph.
Corollary 6. [1] Let G be a graph with connected components $G_i$, for $(1 \leq i \leq l)$. Then the spectrum of G is the union of the spectra of $G_i$ (and multiplicities are added). The same holds for the Laplacian and the signless Laplacian spectrum.

Corollary 7. [1] The multiplicity of 0 as a Laplacian eigenvalue of an undirected graph G equals the number of connected components of G.

Corollary 8. [1] Let the undirected graph G be regular of valency k. Then k is the largest eigenvalue of G and its multiplicity equals the number of connected components of G.

Corollary 9. [1] A graph G is bipartite if and only if the Laplacian spectrum and the signless Laplacian spectrum of G are equal.

Corollary 10. [1] Let G be the complete graph $K_n$ on n vertices. Its spectrum has the $\{(n - 1)^1, (-1)^{n-1} \}$. The Laplacian matrix has spectrum $\{(1, n^{n-1})\}$.

Corollary 11. [1] The spectrum of the complete bipartite graph $K_{m,n}$ is $\{\pm \sqrt{mn}, 0^{m+n-2} \}$. The Laplacian spectrum is $\{0^1, m^{n-1}, n^{m-1}, (m+n)^1\}$.

2 Main Results

We now apply these results to the complete bipartite graph to state and prove our main results. We obtain the bound for complete bipartite graph by making use of the numbers r and s, which are the minimum and maximum degree of the graph.

Theorem 12. Let $G = (V,E)$ be the complete bipartite graph whose vertex set V can be partitioned into two subsets $V_1$ and $V_2$, where $|V_1| = r$ and $|V_2| = s$, for $r \leq s$, and $\lambda_1(G)$ be the Laplacian spectral radius of G. Then

$$\frac{r + \sqrt{r^2 + 8rs}}{2} \leq \lambda_1(G) \leq \frac{s + \sqrt{s^2 + 8rs}}{2}$$ (5)
Moreover, if $G$ is a complete bipartite graph then equality holds and $\lambda_1(G) = 2s$ for $r = s$.

Proof. We first prove the theorem for $\lambda_1(Q)$. Since $Q = D + A$, we have $s_v(Q) = 2d(v)$. Note that $s_v(AD) = s_v(A^2) = \sum_{u \text{adj} v} d(v)$ is true for the complete bipartite graph $G$.

Then

$$s_v(Q^2) = s_v(D(D + A) + AD + A^2)$$
$$= d_v s_v(Q) + 2(\sum_{u \text{adj} v} d(v))$$
$$= d_v s_v(Q) + 2(2m - d(v) - \sum_{u \text{adj} v, u \neq v} d(v))$$
$$\leq ss_v(Q) + 2(2m - s - (r - 1)s)$$
$$= ss_v(Q) + 2(2rs - s - rs + s)$$
$$= ss_v(Q) + 2rs.$$

Hence, for every $v \in V(G)$,

$$s_v(Q^2) - ss_v(Q) - 2rs \leq 0.$$

By theorem 4, $\lambda_1(Q)^2 - s\lambda_1(Q) - 2rs \leq 0$.

Solving the quadratic inequality, we obtain

$$\lambda_1(Q) \leq \frac{s + \sqrt{s^2 + 8rs}}{2}. \quad (6)$$

By theorem 2, we have

$$\lambda_1(G) \leq \frac{s + \sqrt{s^2 + 8rs}}{2}. \quad (7)$$

Similarly, taking the upper bound i.e

$$s_v(Q^2) = s_v(D(D + A) + AD + A^2)$$
$$= d_v s_v(Q) + 2(\sum_{u \text{adj} v} d(v))$$
$$= d_v s_v(Q) + 2(2m - d(v) - \sum_{u \text{adj} v, u \neq v} d(v))$$
$$\geq rs_v(Q) + 2(2m - r - (s - 1)r)$$
$$= rs_v(Q) + 2(2rs - r - (s - 1)r)$$
$$= s_v(Q^2) - rs_v(Q) - 2rs \geq 0.$$

$$s_v(Q^2) - rs_v(Q) \geq 2rs.$$
Solving the quadratic inequality and by theorem 4, we get

\[ \lambda_1(Q) \geq \frac{r + \sqrt{r^2 + 8rs}}{2} \]  
(8)

\[ \lambda_1(G) \geq \frac{r + \sqrt{r^2 + 8rs}}{2} \]  
(9)

Hence, from equations (7) and (9), we get the result.

Consider the Complete bipartite graph \( K_{r,s} \) given in the figure 1, here \( r = 3 \) and \( s = 4 \). So according to theorem 12, the bounds are \( 6.624 \leq \lambda_1(G) \leq 7.2915 \) where the actual calculated value of \( \lambda_1(G) = 7 \).

**Theorem 13.** Let \( G = (V,E) \) be the complete bipartite graph whose vertex set \( V \) can be partitioned into two subsets \( V_1 \) and \( V_2 \), where \( |V_1| = r \) and \( |V_2| = s \), for \( r \leq s \), and \( \lambda_1(G) \) be the Laplacian spectral radius of \( G \). Then the complement \( G^c \) is a disjoint union of \( G_1 \) and \( G_2 \), where \( G_1 \) is the complete graph \( K_r \) and \( G_2 \) is the complete graph \( K_s \). Then \( \lambda_1(G) + \lambda_1(G^c) \leq 3s - 1 \).

**Proof.** The complete bipartite graph \( K_{r,s} \) has \( K_r \) and \( K_s \) as the complement. The Laplacian spectrum of the disjoint union of \( G_i \) components , for \((1 \leq i \leq l)\), is the union of Laplacian spectrum of \( G_1, G_2 \ldots \ldots G_l \).

Let \( \{0 = \alpha_1, \alpha_2 \ldots \ldots \alpha_r \} \) be the spectrum of graph \( G_1 \) and \( \{0 = \beta_1, \beta_2 \ldots \beta_s \} \) be the spectrum of graph \( G_2 \), then spectrum \( G_1 + G_2 \) is the multiset \( \{\alpha_1, \alpha_2 \ldots \ldots \alpha_r, \beta_1, \beta_2 \ldots \beta_s \} \).

Then from [12], we have

\[ \lambda_1(G_1 \cup G_2 \cup \ldots \ldots G_l) = \max \{\lambda_1(G_1) \cup \lambda_1(G_2) \ldots \ldots \lambda_1(G_l)\} \]

Here \( G_1 = K_r \) and \( G_2 = K_s \). Hence, we have

\[ \lambda_1(G^c) = \lambda_1(G_1 \cup G_2) \]

\[ = \max \{(r - 1), (s - 1)\} = (s - 1). \]  
(10)

As \( r \leq s \),

\[ \lambda_1(G) \leq \frac{s + \sqrt{(s^2 + 8rs)}}{2} \leq \frac{s + \sqrt{(s^2 + 8s^2)}}{2} = 2s. \]  
(11)
Hence, \( \lambda_1(G) + \lambda_1(G^c) \leq 2s + (s - 1) = 3s - 1 \).

\[ \Box \]

References


