

## OSCILLATION THEOREMS FOR GENERALIZED SECOND KIND NONLINEAR DELAY DIFFERENCE EQUATIONS

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ABSTRACT. By means of Riccati transformation technique, we establish some new oscillation criteria for generalized second kind nonlinear delay difference equations.

### 1. INTRODUCTION

Difference equations represent a fascinating mathematical area on its own as well as a rich field of the applications in such diverse disciplines as population dynamics, operations research, ecology, economics, biology etc. For general background as difference equations with many examples from diverse fields, one can refer to [1]. The theory of difference equations is based on the operator  $\Delta$  defined as  $\Delta u(k) = u(k+1) - u(k)$ ,  $k \in \mathbb{N}(0) = \{0, 1, 2, \dots\}$ . Eventhough many authors [1], [8] have suggested the definition of  $\Delta$  as

$$(1) \quad \Delta u(k) = u(k+\ell) - u(k), k \in [0, \infty), \ell \in (0, \infty),$$

and no significant progress took place on this line. In [4] they took up the definition of  $\Delta$  as given in (1) and the theory of difference equations are developed in a different direction. For convenience, they labelled the operator  $\Delta$  defined by (1) as  $\Delta_\ell$  and by defining its inverse  $\Delta_\ell^{-1}$ , many interesting results in number theory were obtained. By extending theory of  $\Delta_\ell$  to complex function, some new qualitative properties like rotatory, expanding, shrinking, spiral and weblike were established for the solutions of difference equations involving  $\Delta_\ell$ . The results obtained can be found in [4–6].

Recently, the asymptotic behavior of second order difference equations has been the subject of investigations by many authors,

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see e.g. [1–3, 9–11].

In this paper, we will be concerned with a class of generalized second kind nonlinear delay difference equations

$$(2) \quad \begin{aligned} &\Delta_{\ell_2}(p(k\ell_2 + j_2)\Delta_{\ell_1}(u(k\ell_1 + j_1))^\gamma) \\ &+ q(k\ell_2 + j_2)u((k - \sigma)\ell_1 + j_1) = 0, \end{aligned}$$

when

$$(3) \quad \sum_{k=0}^{\infty} \left( \frac{1}{p(k\ell_2 + j_2)} \right)^{\frac{1}{\gamma}} = \infty$$

holds, where  $j_i = k - \left\lfloor \frac{k_i}{\ell_i} \right\rfloor \ell_i$ ,  $i = 1, 2$  and  $\ell_1, \ell_2 \in (0, \infty)$ , and also assume that the function  $f : [0, \infty) \rightarrow [0, \infty)$  satisfies

$$(4) \quad xf(x) > 0 \text{ and } f(x)/x \geq k > 0 \text{ for } x \neq 0.$$

By a solution of (2) we mean a nontrivial function  $u(k\ell_1 + j_1)$  which is defined for  $k \geq -\sigma$ , and satisfies equation (2) for  $k \in [0, \infty)$ . Clearly if

$$(5) \quad u(k\ell_1 + j_1) = A(k\ell_1 + j_1) \text{ for } k \in [-\sigma, 0]$$

are given, then equation (2) has a unique solution satisfying the initial condition (5). A solution  $u(k)$  of (2) is said to be oscillatory if for every  $k_1 > 0$  there exists an  $k \geq k_1$  such that  $u(k)u(k+\ell) \leq 0$ , otherwise it is nonoscillatory. Equation (2) is said to be oscillatory if all its solutions are oscillatory.

In this paper, by using the Riccati transformation technique, we present some new oscillation criteria for equation (2) when (3) holds.

## 2. PRELIMINARIES

**Definition 1.** An equation is of the form

$$f(k, \Delta_{\ell_1}u(k), \Delta_{\ell_1, \ell_2}u(k), \dots, \Delta_{\ell_1, \ell_2, \dots, \ell_n}u(k)) = 0$$

is called the generalized  $n^{th}$ -kind difference equation.

**Definition 2.**

$$(6) \quad u(k) - [u(k + \ell_1) + v(k + \ell_2)] + u(k + \ell_1 + \ell_2) = v(k)$$

is the generalized nonhomogeneous second kind difference equation and by taking  $v(k) = 0$  in (6), then the equation (6) is called generalized homogenous second kind difference equation.

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- Remark 3.* (1) In (6), taking  $\ell_1 = \ell_2 = \ell$ , then the equation (6) is called generalized second order difference equation.  
 (2) In (6), substituting  $\ell_1 = \ell_2 = 1$ , then the equation (6) is called second order difference equation.

**Definition 4.** [4] For  $\lambda \in N(1)$ , the generalized polynomial factorial is defined by

$$(7) \quad k_\ell^{(\lambda)} = k(k - \ell)(k - 2\ell) \cdots (k - (\lambda - 1)\ell).$$

## 3. MAIN RESULTS

**Theorem 5.** Assume that (3) and (4) hold. Furthermore, assume that there exists a positive real valued function  $\rho(k\ell_2 + j_2)$  such that for every positive constant  $M$ ,

$$(8) \quad \limsup_{k \rightarrow \infty} \sum_{r=k_0}^k [n\rho(r\ell_1 + j_1)q(r\ell_1 + j_1) - \frac{(p((r - \sigma)\ell_2 + j_2))^{\frac{1}{\gamma}} p(k\ell_2 + \ell_1 + j_2)(\Delta_{\ell_1}\rho(r\ell_1 + j_1))^2}{4(M)^{\frac{\gamma-1}{\gamma}} p((k+1)\ell_2 + j_2)\rho(r\ell_1 + j_1)}] = \infty$$

for some  $k_0 > 0$ . Then every solution of equation (2) oscillates.

*Proof.* Suppose the contrary that  $u(k\ell_1 + j_1)$  is a nonoscillatory solution of (2). Without loss of generality, we may assume that  $u(k\ell_1 + j_1)$  is an eventually positive solution of (2) such that  $u(k\ell_1 + j_1) > 0$  and  $u((k - \sigma)\ell_1 + j_1) > 0$  for all  $k \geq k_0 > 0$ . We shall consider only this case, since the substitution  $v(k\ell_1 + j_1) = -u(k\ell_1 + j_1)$  transforms equation (2) into an equation of the same form. In view of (2) we have for  $k \geq k_0$ ,

$$(9) \quad \begin{aligned} & \Delta_{\ell_2}(p(k\ell_2 + j_2)(\Delta_{\ell_1}u(k\ell_1 + j_1))^\gamma) \\ & = -q(k\ell_2 + j_2)f(u((k - \sigma)\ell_1 + j_1)) \leq 0, \end{aligned}$$

and so  $p(k\ell_2 + j_2)(\Delta_{\ell_1}u(k\ell_1 + j_1))^\gamma$  is an eventually nonincreasing function. First we show that  $p(k\ell_2 + j_2)(\Delta_{\ell_1}u(k\ell_1 + j_1))^\gamma$  is eventually positive. Indeed, since  $q(k\ell_2 + j_2)$  has a positive subsequence, the nondecreasing sequence  $p(k\ell_2 + j_2)(\Delta_{\ell_1}u(k\ell_1 + j_1))^\gamma$  is either eventually positive or eventually negative. Suppose there exists real number  $k_1\ell_1 + j_1 \geq 0$  such that  $p(k_1\ell_2 + j_2)(\Delta_{\ell_1}u(k_1\ell_1 + j_1))^\gamma = c < 0$ . Then, from (9) we have

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$p(k\ell_2 + j_2)(\Delta_{\ell_1} u(k\ell_1 + j_1))^\gamma \leq p(k_1\ell_2 + j_2)(\Delta_{\ell_1} u(k_1\ell_1 + j_1))^\gamma = c$   
for  $k \geq k_1$ , so that

$$\Delta_{\ell_1} u(k\ell_1 + j_1) \leq c^{\frac{1}{\gamma}} \left( \frac{1}{p(k\ell_2 + j_2)} \right)^{1/\gamma},$$

which implies that, as  $k \rightarrow \infty$ ,

$$u(k\ell_1 + j_1) \leq u(k_1\ell_1 + j_1) + c^{\frac{1}{\gamma}} \sum_{i=0}^{k-1} \left( \frac{1}{p(k_1\ell_2 + j_2 + i\ell_1)} \right)^{1/\gamma} \rightarrow -\infty,$$

which contradicts the fact that  $u(k\ell_1 + j_1) > 0$  for all large  $k$ . Hence  $p(k\ell_2 + j_2)(\Delta_{\ell_1} u(k\ell_1 + j_1))^\gamma$  is eventually positive. Therefore, we see that there is some  $k_0$  such that

$$(10) \quad \begin{aligned} u(k\ell_1 + j_1) &> 0, \quad \Delta_{\ell_1} u(k\ell_1 + j_1) \geq 0, \\ \Delta_{\ell_2}(p(k\ell_2 + j_2)(\Delta_{\ell_1} u(k\ell_1 + j_1))^\gamma) &\leq 0, \quad k \geq k_0. \end{aligned}$$

Define the function  $z(k\ell_1 + j_1)$  by the Riccati transformation

$$(11) \quad z(k\ell_1 + j_1) = \rho(k\ell_1 + j_1) \frac{p(k\ell_2 + j_2)(\Delta_{\ell_1} u(k\ell_1 + j_1))^\gamma}{u(k\ell_1 - \sigma\ell_2 + j_1)}.$$

We have  $z(k\ell_2 + j_2) > 0$  and

$$(12) \quad \begin{aligned} &\Delta_{\ell_1} z(k\ell_1 + j_1) \\ &= p(k\ell_2 + k\ell_1 + j_2)(\Delta_{\ell_1} u((k+1)\ell_1 + j_1))^\gamma \Delta_{\ell_1} \left[ \frac{\rho(k\ell_1 + j_1)}{u(k\ell_1 - \sigma\ell_2 + j_1)} \right] \\ &+ \frac{\rho(k\ell_1 + j_1) \Delta_{\ell_1}(p(k\ell_2 + j_2)(\Delta_{\ell_1} u(k\ell_1 + j_1))^\gamma)}{u(k\ell_1 - \sigma\ell_2 + j_1)}. \end{aligned}$$

Now, from (2), (4) and (12), we find

$$(13) \quad \begin{aligned} \Delta_{\ell_1} z(k\ell_1 + j_1) &\leq -n\rho(k\ell_1 + j_1)q(k\ell_1 + j_1) \\ &+ \frac{\Delta_{\ell_1}\rho(k\ell_1 + j_1)}{\rho((k+1)\ell_1 + j_1)} z((k+1)\ell_1 + j_1) \\ &- \left[ \frac{\rho(k\ell_1 + j_1)p(k\ell_2 + \ell_1 + j_2)}{u((k+1)\ell_1 - \sigma\ell_2 + j_1)u(k\ell_1 - \sigma\ell_2 + j_1)} \right. \\ &\left. (\Delta_{\ell_1} u(k\ell_1 - \sigma\ell_2 + j_1))(\Delta_{\ell_1} u((k+1)\ell_1 + j_1))^\gamma \right]. \end{aligned}$$

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But, from (10), we have for some  $k \geq k_1 = k_0 + \sigma\ell_1$

$$\begin{aligned} & p((k - \sigma)\ell_2 + j_2)(\Delta_{\ell_1} u(k\ell_1 - \sigma\ell_2 + j_1))^\gamma \\ & \geq p((k + 1)\ell_2 + j_2)(\Delta_{\ell_1} u(k\ell_1 + \ell_2 + j_1))^\gamma, \\ (14) \quad & \text{and } u((k + 1)\ell_1 - \sigma\ell_2 + j_1) \geq u(k\ell_1 - \sigma\ell_2 + j_1), \end{aligned}$$

and thus (13) and (14) imply that

$$\begin{aligned} & \Delta_{\ell_1} z(k\ell_1 + j_1) \leq -n\rho(k\ell_1 + j_1)q(k\ell_1 + j_1) \\ & + \frac{z((k + 1)\ell_1 + j_1)\Delta_{\ell_1}\rho(k\ell_1 + j_1)}{\rho((k + 1)\ell_1 + j_1)} \\ & - \left[ \left( \frac{p((k + 1)\ell_2 + j_2)}{p((k - \sigma)\ell_2 + j_2)} \right)^{\frac{1}{\gamma}} \rho(k\ell_1 + j_1)p(k\ell_2 + \ell_1 + j_2) \right. \\ (15) \quad & \left. \frac{(\Delta_{\ell_1} u((k + 1)\ell_1 + j_1))^\gamma (\Delta_{\ell_1} u(k\ell_1 + \ell_2 + j_1))}{(u((k + 1)\ell_1 - \sigma\ell_2 + j_1))^2} \right] \end{aligned}$$

hence, from (11) and (15), we obtain

$$\begin{aligned} & \Delta_{\ell_1} z(k\ell_1 + j_1) \leq -n\rho(k\ell_1 + j_1)q(k\ell_1 + j_1) \\ & + \frac{z((k + 1)\ell_1 + j_1)\Delta_{\ell_1}\rho(k\ell_1 + j_1)}{\rho((k + 1)\ell_1 + j_1)} - \left[ \left( \frac{p((k + 1)\ell_2 + j_2)}{p((k - \sigma)\ell_2 + j_2)} \right)^{\frac{1}{\gamma}} \right. \\ (16) \quad & \left. \frac{\rho(k\ell_1 + j_1)}{(\rho((k + 1)\ell_1 + j_1))^2 p(k\ell_2 + \ell_1 + j_2)} \frac{z^2((k + 1)\ell_1 + j_1)}{(\Delta_{\ell_1} u(k\ell_1 + \ell_2 + j_1))^{\gamma-1}} \right]. \end{aligned}$$

Now, due to the fact that  $p(k\ell_2 + j_2)(\Delta_{\ell_1} u(k\ell_1 + j_1))^\gamma$  is a positive and nonincreasing function, there exists an  $k_2 \geq k_1$  sufficiently large such that  $p(k\ell_2 + j_2)(\Delta_{\ell_1} u(k\ell_1 + j_1))^\gamma \leq 1/M$  for some positive constant  $M$  and  $k \geq k_2$ , and hence by (10) we have  $p((k + 1)\ell_2 + j_2)(\Delta_{\ell_1} u(k\ell_1 + \ell_2 + j_1))^\gamma \leq 1/M$ , so that

$$(17) \quad \frac{1}{(\Delta_{\ell_1} u(k\ell_1 + \ell_2 + j_1))^{\gamma-1}} \geq (Mp((k + 1)\ell_2 + j_2))^{\frac{\gamma-1}{\gamma}}.$$

Substitute (17) into (16), we obtain

$$\begin{aligned} & \Delta_{\ell_1} z(k\ell_1 + j_1) \leq -n\rho(k\ell_1 + j_1)q(k\ell_1 + j_1) \\ & + \frac{z((k + 1)\ell_1 + j_1)\Delta_{\ell_1}\rho(k\ell_1 + j_1)}{\rho((k + 1)\ell_1 + j_1)} \\ (18) \quad & - \frac{(M)^{\frac{\gamma-1}{\gamma}} p((k + 1)\ell_2 + j_2)\rho(k\ell_1 + j_1)z^2((k + 1)\ell_1 + j_1)}{p(k\ell_2 + \ell_1 + j_1)(p((k - \sigma)\ell_2 + j_2))^{\frac{1}{\gamma}}(\rho((k + 1)\ell_1 + j_1))^2}. \end{aligned}$$

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then

$$\begin{aligned}
& \Delta_{\ell_1} z(k\ell_1 + j_1) \leq -n\rho(k\ell_1 + j_1)q(k\ell_1 + j_1) \\
& + \frac{p(k\ell_2 + \ell_1 + j_2)(p((k - \sigma)\ell_2 + j_2))^{\frac{1}{\gamma}}(\Delta_{\ell_1}\rho(k\ell_1 + j_1))^2}{4(M)^{\frac{\gamma-1}{\gamma}}p((k+1)\ell_2 + j_2)\rho(k\ell_1 + j_1)} \\
& - \left[ \frac{\sqrt{(M)^{\frac{\gamma-1}{\gamma}}p((k+1)\ell_2 + j_2)\rho(k\ell_1 + j_1)z((k+1)\ell_1 + j_1)}}{\rho((k+1)\ell_1 + j_1)\sqrt{p(k\ell_2 + \ell_1 + j_2)(p((k - \sigma)\ell_2 + j_2))^{\frac{1}{\gamma}}}} \right. \\
& \left. - \frac{\sqrt{p(k\ell_2 + \ell_1 + j_2)(p((k - \sigma)\ell_2 + j_2))^{\frac{1}{\gamma}}\Delta_{\ell_1}\rho(k\ell_1 + j_1)}}{2\sqrt{(M)^{\frac{\gamma-1}{\gamma}}p((k+1)\ell_2 + j_2)\rho(k\ell_1 + j_1)}} \right]^2 \\
& < -[n\rho(k\ell_1 + j_1)q(k\ell_1 + j_1) \\
(19) \quad & - \frac{p(k\ell_2 + \ell_1 + j_2)(p((k - \sigma)\ell_2 + j_2))^{\frac{1}{\gamma}}(\Delta_{\ell_1}\rho(k\ell_1 + j_1))^2}{4(M)^{\frac{\gamma-1}{\gamma}}p((k+1)\ell_2 + j_2)\rho(k\ell_1 + j_1)}] .
\end{aligned}$$

Summing (19) from  $k_2$  to  $k$ , we obtain

$$\begin{aligned}
& -z(k_2\ell_1 + j_1) < z((k+1)\ell_1 + j_1) - z(k_2\ell_1 + j_1) \\
& < -\sum_{r=k_2}^k [n\rho(r\ell_1 + j_1)q(r\ell_1 + j_1) \\
(20) \quad & - \frac{p(r\ell_2 + \ell_1 + j_2)(p((r - \sigma)\ell_2 + j_2))^{\frac{1}{\gamma}}(\Delta_{\ell_1}\rho(r\ell_1 + j_1))^2}{4(M)^{\frac{\gamma-1}{\gamma}}p((r+1)\ell_2 + j_2)\rho(r\ell_1 + j_1)}]
\end{aligned}$$

which yields

$$\begin{aligned}
& \sum_{r=k_2}^k [n\rho(r\ell_1 + j_1)q(r\ell_1 + j_1) \\
(21) \quad & - \frac{p(r\ell_2 + \ell_1 + j_2)(p((r - \sigma)\ell_2 + j_2))^{\frac{1}{\gamma}}(\Delta_{\ell_1}\rho(r\ell_1 + j_1))^2}{4(M)^{\frac{\gamma-1}{\gamma}}p((r+1)\ell_2 + j_2)\rho(r\ell_1 + j_1)}] < c_1,
\end{aligned}$$

for all large  $k$ , and this is contrary to (8). The proof is complete.  $\square$ 

From Theorem 5, we can obtain different conditions for oscillation of all solutions of (2) by different choices of  $\rho(k)$ . When  $\gamma = 1$ , equation (2) reduces to the nonlinear delay difference equation and then Theorem 5 reduces to the following Corollary.

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**Corollary 6.** Assume that (3) and (4) hold and there exists a positive sequence  $q(k\ell_2 + j_2)$  such that

$$(22) \quad \limsup_{k \rightarrow \infty} \sum_{r=k_0}^k \left[ n\rho(r\ell_1 + j_1)q(r\ell_1 + j_1) - \frac{p(r\ell_2 + \ell_1 + j_2)(p(r - \sigma)\ell_2 + j_2)(\Delta_{\ell_1}\rho(r\ell_1 + j_1))^2}{4p((r + 1)\ell_2 + j_2)\rho(r\ell_1 + j_1)} \right] = \infty.$$

Then every solution of equation (2) oscillates.

**Example 7.** Consider the second kind discrete Euler equation

$$(23) \quad \Delta_{\ell_2}\Delta_{\ell_1}u(k\ell_1 + j_1) + \frac{\mu}{(k\ell_2 + j_2)^2}u(k\ell_1 + j_1) = 0, \quad k \geq 1,$$

where  $\mu > \frac{1}{4}$ . Then,  $p(k\ell_1 + j_1) = 1$ ,  $n = 1$  and  $\gamma = 1$ . If we take  $\rho(k\ell_1 + j_1) = k\ell_1 + j_1$ , then we have

$$\begin{aligned} & \sum_{s=k_0}^k \left[ (s\ell_1 + j_1)q(s\ell_1 + j_1) - \frac{(\Delta_{\ell_1}(s\ell_1 + j_1))^2}{4(s\ell_1 + j_1)} \right] \\ &= \sum_{s=1}^k \left[ \frac{\mu(s\ell_1 + j_1)}{(s\ell_1 + j_1)^2} - \frac{\ell_1^2}{4(s\ell_1 + j_1)} \right] = \sum_{s=1}^k \frac{4\mu - \ell_1^2}{4(s\ell_1 + j_1)} \rightarrow \infty. \end{aligned}$$

as  $k \rightarrow \infty$ . By Corollary 6, every solution of (23) oscillates. It is known that when  $\mu \leq \frac{1}{4}$ , (23) has a nonoscillatory solution. Hence, Theorem 5 and Corollary 6 are sharp.

**Example 8.** Consider the second kind nonlinear delay difference equation

$$(24) \quad \Delta_{\ell_2} \left( \frac{(k+1)\ell_2 + j_2}{(k+2)\ell_2 + j_2} \Delta_{\ell_1}u(k\ell_1 + j_1) \right) + \frac{\mu}{(k\ell_2 + j_2)^2}u((k-1)\ell_2 + j_2)(1 + u^2((k-1)\ell_1 + j_1)) = 0,$$

where  $k \geq 1$ ,  $\mu > \frac{1}{4}$ ,  $f(x)/x \geq n = 1$ ,  $\sigma = 1$  and  $\gamma = 1$ . If we take  $\rho(k\ell_1 + j_1) = k\ell_1 + j_1$ , then we have

$$\begin{aligned} & \sum_{s=k_0}^k \left[ \frac{\mu(s\ell_1 + j_1)}{(s\ell_1 + j_1)^2} - \frac{s\ell_1 + j_1}{(s+1)\ell_1 + j_1} \frac{(\Delta_{\ell_1}(s\ell_1 + j_1))^2}{4(s\ell_1 + j_1)} \right] \\ &= \sum_{s=1}^k \left[ \frac{\mu}{s\ell_1 + j_1} - \frac{\ell_1^2}{4((s+1)\ell_1 + j_1)} \right] \geq \sum_{s=1}^k \left[ \frac{4\mu - \ell_1^2}{4(s\ell_1 + j_1)} \right] \rightarrow \infty \end{aligned}$$

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as  $k \rightarrow \infty$ . Thus Theorem 5 asserts that every solution of (24) is oscillatory when  $\mu > \frac{1}{4}$ .

**Theorem 9.** Assume that (3) and (4) hold. Let  $\rho(k\ell_1 + j_1)$  be a positive function. Furthermore, we assume that there exists a double function  $\{H(m, k\ell_1 + j_1) : m \geq k \geq 0\}$  such that (i)  $H(m, m) = 0$  for  $m \geq 0$ , (ii)  $H(m, k\ell_1 + j_1) > 0$  for  $m > k\ell_1 + j_1 \geq 0$  (iii)  $\Delta_{2(\ell_1)}H(m, k\ell_1 + j_1) = H(m, ((k+1)\ell_1 + j_1)) - H(m, k\ell_1 + j_1) \leq 0$  for  $m \geq k\ell_1 + j_1 \geq 0$ . If

$$(25) \quad \limsup_{m \rightarrow \infty} \frac{1}{H(m, 0)} \sum_{k=k_0}^{m-1} \left[ nH(m, k\ell_2 + j_2)\rho(k\ell_2 + j_2)q((k+1)\ell_2 + j_2) - \{(\rho((k+1)\ell_2 + j_2))^2(p((k-\sigma)\ell_2 + j_2))^{\frac{1}{\gamma}} \right. \\ \left. \frac{\left( h(m, k\ell_2 + j_2) - \frac{\Delta_{\ell_2}\rho(k\ell_2 + j_2)\sqrt{H(m, k\ell_2 + j_2)}}{\rho((k+1)\ell_2 + j_2)} \right)^2}{4(M)^{\frac{\gamma-1}{\gamma}}\rho(k\ell_2 + j_2)} \right] < \infty$$

where  $h(m, k\ell_2 + j_2) = -\frac{\Delta_{2(\ell_2)}H(m, k\ell_2 + j_2)}{\sqrt{H(m, k\ell_2 + j_2)}}$ ,  $m > k \geq 0$ , for every positive number  $M$ , then every solution of equation (2) oscillates.

*Proof.* We proceed as in the proof of Theorem 5. We may assume that (2) has a nonoscillatory solution  $u(k\ell + j)$  such that  $u(k\ell_2 + j_2) > 0$  and  $u((k-\sigma)\ell_2 + j_2) > 0$ . From the proof of Theorem 5 we see that (10) holds for  $k \geq k_1$ . Defining again  $z(k\ell_1 + j_1)$  by the Riccati transformation (11), we have  $z(k\ell_1 + j_1) > 0$  and there is some  $M > 0$  such that (18) holds for  $k \geq k_2$ . For the sake of convenience, let us set

$$\overline{\rho(k\ell_1 + j_1)} = \frac{M^{\frac{(\gamma-1)}{\gamma}} p(k\ell_2 + \ell_1 + j_2)\rho(k\ell_1 + j_1)}{p((k+1)\ell_2 + j_2)p^{\frac{1}{\gamma}}((k-\sigma)\ell_2 + j_2)},$$

then, from (18) we have for  $k \geq k_2$ ,

$$n\rho(k\ell_1 + j_1)q(k\ell_1 + j_1) \leq \frac{z(k\ell_1 + j_1)\Delta_{\ell_1}\rho(k\ell_1 + j_1)}{\rho((k+1)\ell_1 + j_1)} \\ - \Delta_{\ell_1}z(k\ell_1 + j_1) - \frac{\overline{\rho(k\ell_1 + j_1)}}{\rho^2((k+1)\ell_1 + j_1)}z^2((k+1)\ell_1 + j_1).$$

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Therefore, for  $k \geq k_2$ ,

$$\begin{aligned}
& \sum_{k=k_2}^{m-1} nH(m, k\ell_1 + j_1)\rho(k\ell_1 + j_1)q(k\ell_1 + j_1) \\
& \leq - \sum_{k=k_2}^{m-1} H(m, k\ell_1 + j_1)\Delta_{\ell_1}z(k\ell_1 + j_1) \\
& + \sum_{k=k_2}^{m-1} H(m, k\ell_1 + j_1)\frac{\Delta_{\ell_1}\rho(k\ell_1 + j_1)}{\rho((k+1)\ell_1 + j_1)}z((k+1)\ell_1 + j_1) \\
& - \sum_{k=k_2}^{m-1} H(m, k\ell_1 + j_1)\frac{\overline{\rho(k\ell_1 + j_1)}}{\rho^2((k+1)\ell_1 + j_1)}z^2((k+1)\ell_1 + j_1),
\end{aligned}$$

which yields, after summing by parts,

$$\begin{aligned}
& \sum_{k=k_2}^{m-1} nH(m, k\ell_1 + j_1)\rho(k\ell_1 + j_1)q(k\ell_1 + j_1) \leq H(m, k_2\ell_1 + j_1)z(k_2\ell_1 + j_1) \\
& + \sum_{k=k_2}^{m-1} z((k+1)\ell_1 + j_1)\Delta_{2(\ell_1)}H(m, k\ell_1 + j_1) \\
& + \sum_{k=k_2}^{m-1} \frac{H(m, k\ell_1 + j_1)\Delta_{\ell_1}\rho(k\ell_1 + j_1)z((k+1)\ell_1 + j_1)}{\rho((k+1)\ell_1 + j_1)} \\
& - \sum_{k=k_2}^{m-1} H(m, k\ell_1 + j_1)\frac{\overline{\rho(k\ell_1 + j_1)}}{\rho^2((k+1)\ell_1 + j_1)}z^2((k+1)\ell_1 + j_1) \\
& = H(m, k_2\ell_1 + j_1)z(k_2\ell_1 + j_1) \\
& - \sum_{k=k_2}^{m-1} h(m, k\ell_1 + j_1)\sqrt{H(m, k\ell_1 + j_1)}z((k+1)\ell_1 + j_1) \\
& + \sum_{k=k_2}^{m-1} H(m, k\ell_1 + j_1)\frac{\Delta_{\ell_1}\rho(k\ell_1 + j_1)}{\rho((k+1)\ell_1 + j_1)}z((k+1)\ell_1 + j_1) \\
& - \sum_{k=k_2}^{m-1} H(m, k\ell_1 + j_1)\frac{\overline{\rho(k\ell_1 + j_1)}}{\rho^2((k+1)\ell_1 + j_1)}z^2((k+1)\ell_1 + j_1)
\end{aligned}$$

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$$\begin{aligned}
&= H(m, k_2 \ell_1 + j_1) z(k_2 \ell_1 + j_1) \\
&\quad - \sum_{k=k_2}^{m-1} \left[ \frac{\sqrt{H(m, k \ell_1 + j_1) \rho(k \ell_1 + j_1)}}{\rho((k+1) \ell_1 + j_1)} z((k+1) \ell_1 + j_1) \right. \\
&\quad \left. + \frac{\rho((k+1) \ell_1 + j_1) \left( h(m, k \ell_1 + j_1) \sqrt{H(m, k \ell_1 + j_1)} - \frac{\Delta_{\ell_1} \rho(k \ell_1 + j_1) H(m, k \ell_1 + j_1)}{\rho((k+1) \ell_1 + j_1)} \right)}{2 \sqrt{H(m, k \ell_1 + j_1) \rho(k \ell_1 + j_1)}} \right]^2 \\
&\quad + \frac{1}{4} \sum_{k=k_2}^{m-1} \frac{(\rho((k+1) \ell_1 + j_1))^2 \left( h(m, k \ell_1 + j_1) - \frac{\Delta_{\ell_1} \rho(k \ell_1 + j_1) \sqrt{H(m, k \ell_1 + j_1)}}{\rho((k+1) \ell_1 + j_1)} \right)^2}{\rho(k \ell_1 + j_1)}.
\end{aligned}$$

Then,

$$\begin{aligned}
&\sum_{k=k_2}^{m-1} [nH(m, k \ell_1 + j_1) \rho(k \ell_1 + j_1) q((k+1) \ell_1 + j_1) \\
&\quad - \frac{(\rho((k+1) \ell_1 + j_1))^2 \left( h(m, k \ell_1 + j_1) - \frac{\Delta_{\ell_1} \rho(k \ell_1 + j_1) \sqrt{H(m, k \ell_1 + j_1)}}{\rho((k+1) \ell_1 + j_1)} \right)^2}{4 \rho(k \ell_1 + j_1)}] \\
&< H(m, k_2 \ell_1 + j_1) z(k_2 \ell_1 + j_1) \leq H(m, 0) z(k_2 \ell_1 + j_1),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\sum_{k=0}^{m-1} [nH(m, k \ell_1 + j_1) \rho(k \ell_1 + j_1) q(k \ell_1 + j_1) \\
&\quad - \frac{(\rho((k+1) \ell_1 + j_1))^2 \left( h(m, k \ell_1 + j_1) - \frac{\Delta_{\ell_1} \rho(k \ell_1 + j_1) \sqrt{H(m, k \ell_1 + j_1)}}{\rho((k+1) \ell_1 + j_1)} \right)^2}{4 \rho(k \ell_1 + j_1)}] \\
&< nH(m, 0) \sum_{k=0}^{k_2-1} \rho(k \ell_1 + j_1) q(k \ell_1 + j_1) + H(m, 0) z(k_1 \ell_1 + j_1).
\end{aligned}$$

Hence

$$\begin{aligned}
&\limsup_{m \rightarrow \infty} \frac{1}{H(m, 0)} \sum_{k=0}^{m-1} [nH(m, k \ell_1 + j_1) \rho(k \ell_1 + j_1) q(k \ell_1 + j_1) \\
&\quad - \frac{(\rho((k+1) \ell_1 + j_1))^2 \left( h(m, k \ell_1 + j_1) - \frac{\Delta_{\ell_1} \rho(k \ell_1 + j_1) \sqrt{H(m, k \ell_1 + j_1)}}{\rho((k+1) \ell_1 + j_1)} \right)^2}{4 \rho(k \ell_1 + j_1)}] < \infty,
\end{aligned}$$

which is contrary to (25). The proof is complete.  $\square$

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