

Connected Regular Domination number of $\Gamma(Z_n)$

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August 9, 2017

Abstract

The connected dominating set was first suggested by S. T. Hedetniemi and elaborate treatment of this parameter appears in E. Sampathkumar and H. B. Walikar([?]). The concepts of regular domination was first introduced by prof. E. Sampathkumar. On combining these two parameters we obtain a new parameter namely connected regular domination as, let G be a graph, V is a vertex set of G and $D \subseteq V$, then D is said to be connected regular domination, if it satisfies the following conditions, (i) D is Dominating set (ii) D is regular (iii) D is connected. The first simplification of Beck's ([4]) zero divisor graph was introduced by D. F. Anderson and P. S. Livingston ([2]). Their motivation was to give a better illustration of the zero divisor structure of the ring. In this paper, we find the connected regular domination number of $\Gamma(Z_n)$, for some cases of n .

AMS Subject Classification: 05C25, 05C69

Key Words and Phrases: Connected domination, Regular domination, Zero divisor graph.

1 Introduction

The connected dominating set was first suggested by S. T. Hedetniemi and elaborate treatment of this parameter appears in E. Sampathkumar and H. B. Walikar. The concepts of regular domination was first introduced by prof. E. Sampathkumar. A recent book on the topic of domination [2] lists over 1,200 papers related to domination in graphs, and several hundred papers on the topic have been written since the publication of the book few years ago. The first simplification of Beck's ([4]) zero divisor graph was introduced by D. F. Anderson and P. S. Livingston ([2]). Their motivation was to give a better illustration of the zero divisor structure of the ring. In this new zero divisor graph, which is still a simple graph with edges defined the same way as above, only the zero divisors of the ring are included. That is, non-zero elements r of R such that $\text{Ann}_R(r) \neq 0$. Given a ring R , let $Z^*(R)$ denote the set of zero divisors of R . Let $Z^*(R)$ denote the graph whose vertex set is $Z^*(R)$, such that distinct vertices r and s are adjacent provided that $rs = 0$. Anderson and Livingston often focus on the case when R is finite, as these rings yield finite graphs. D. F. Anderson and P. S. Livingston, and others, e.g., ([1], [3], [5], investigate the interplay between the graph theoretic properties of $\Gamma(R)$ and the ring theoretic properties of R .

Definition 1. A **dominating set** is a set of vertices such that each vertex of V is either in D or has atleast one neighbour in D . The minimum cardinality of such a set is called the domination number of G denoted by $\gamma(G)$.

Definition 2. A **connected dominating set** D is a set of vertices of a graph G such that every vertex in $V - D$ is adjacent to atleast one vertex in D and the subgraph $\langle D \rangle$ induced by the set D is connected. The connected domination number $\gamma_c(G)$ is the minimum cardinality of the connected dominating sets of G .

Definition 3. A **regular dominating** set is a dominating set D of $V(G)$ if $\langle D \rangle$ is regular. The minimum cardinality of a regular dominating set is called regular domination number of G and is denoted by $\gamma_r(G)$.

Corollary 4. A graph G has a connected dominating set if and only if G is connected.[7]

Theorem 5. A dominating set D of a connected graph G is a minimal dominating set if and only if for each vertex in $u \in D$, one of two conditions holds: i) u is not adjacent to any vertex in D . ii) there exists a vertex $v \in V - D$ for which $N(v) \cap D = u$. [7]

Corollary 6. Since any maximal independent set is a regular dominating set, the existence of a regular dominating set is always guaranteed in any graph. Also for any graph G , $\gamma_r(G) = 1$ if and only if G has a full degree vertex.

Theorem 7. A graph $\Gamma(Z_n)$ has a connected domination set if and only if $\Gamma(Z_n)$ is connected and n is a composite number.

On combining the two parameters we obtain a new parameter namely connected regular domination.

Definition 8. Let G be a graph, V is a vertex set of G and $D \subseteq V$, then D is said to be **Connected Regular dominating**, if it satisfies the following conditions, (i) D is dominating set (ii) D is regular (iii) D is connected. The minimum cardinality of connected regular domination set is denoted by $\gamma_{cr}(G)$

2 Connected Regular domination of $\Gamma(Z_n)$

For any graph $\Gamma(Z_n)$, it is clear that the connected regular domination numbers lies in the range $1 \leq \gamma_{cr}(\Gamma(Z_n)) \leq n$. Hence, there are only finite number of possible minimum cardinality connected regular dominating sets of $\Gamma(Z_n)$. In this section we compute the connected regular domination number of $\Gamma(Z_n)$ with minimum cardinalities.

Definition 9. Let $\Gamma(Z_n)$ be a graph, V is a vertex set of G and $D \subseteq V$, then D is said to be **Connected Regular dominating**

set, if it satisfies the following conditions, (i) D is dominating set (ii) D is regular (iii) D is connected. The minimum cardinality of connected regular domination set is denoted by $\gamma_{cr}\Gamma(Z_n)$.

3 Connected Regular domination number of complete and complete bipartite graph of $\Gamma(Z_n)$

In this section we evaluate the connected regular domination number of complete and complete bipartite graph of $\Gamma(Z_n)$.

Theorem 10. *For any graph $\Gamma(Z_{p^2})$, where $p > 2$ is any prime number then $\gamma_{cr}(\Gamma(Z_{p^2})) = 1$.*

Proof. The vertex set of $\Gamma(Z_{p^2})$ is $\{p, 2p, 3p, \dots, p(p-1)\}$. The total number of vertices in $\Gamma(Z_{p^2})$ is $p-1$. Any two vertices of V are adjacent in $\Gamma(Z_{p^2})$. Clearly $\Gamma(Z_{p^2})$ is a complete graph K_{p-1} . Then any one vertex of V will dominate the remaining vertices in $\Gamma(Z_{p^2})$. Thus $\gamma(\Gamma(Z_{p^2}))$ is one. That dominating vertex is itself connected and regular. Thus, $\gamma_{cr}(\Gamma(Z_{p^2})) = 1$. $\square$

Theorem 11. *For any graph $\Gamma(Z_{2p})$ with p vertices and maximum vertex degree $\Delta(\Gamma(Z_{2p}))$ then $\gamma_{cr}(\Gamma(Z_{2p})) = p - \Delta(\Gamma(Z_{2p}))$, if and only if $\Gamma(Z_{2p})$ is a star graph.*

Proof. Consider $\Gamma(Z_{2p})$ as a star graph. Let the maximum degree $\Delta(\Gamma(Z_{2p}))$ be vertex v . By considering v as the root, $\Gamma(Z_{2p})$ has exactly $\Delta(\Gamma(Z_{2p}))$ branches from v . Since the vertices in each of these branches has a degree less than 3. Thus the number of leaves in $\Gamma(Z_{2p})$ is exactly $\Delta(\Gamma(Z_{2p}))$. By using the above theorem, the connected regular domination of $\Gamma(Z_{2p}) = 1$. That is, $1 = \text{number of points} - \text{maximum degree} = p - (p-1) = 1$. Hence $\gamma_{cr}(\Gamma(Z_{2p})) = p - \Delta(\Gamma(Z_{2p}))$.

Conversely, if $\Gamma(Z_{2p})$ is not a star graph, then there exists a vertex other than v with degree not less than 3 in $\Gamma(Z_{2p})$. Therefore $\Gamma(Z_{2p})$ has a branch with more than one leaf in it. This shows that $\Gamma(Z_{2p})$ has more than $\Delta(\Gamma(Z_{2p}))$ leaves, which is a contradiction and hence the theorem. $\square$

Theorem 12. For any graph $\Gamma(Z_{2p})$, where $p > 2$ is any prime number then $\gamma_{cr}(\Gamma(Z_{2p})) = 1$.

Proof. The vertex set V of $\Gamma(Z_{2p})$ is $\{2, 4, 6, \dots, 2(p-1), p\}$. The total number of vertices in $\Gamma(Z_{2p})$ is p . Let u and v be any two vertices from $V - p$. Then clearly $2p$ does not divide uv . Thus there exist no edges between the vertices in $V - p$. Similarly, let u be any vertex from $\{2, 4, 6, \dots, 2(p-1)\}$ and $v = p$. Then clearly $2p$ must divide uv . Note that p is adjacent to all remaining vertices in V . Thus the $\gamma(\Gamma(Z_{2p})) = 1$. Also the vertex p is both connected and regular. Thus $\gamma_{cr}(\Gamma(Z_{2p})) = 1$. $\square$

Theorem 13. For any graph $\Gamma(Z_{3p})$, where $p > 2$ is any prime number then $\gamma_{cr}(\Gamma(Z_{3p})) = 2$.

Proof. The vertex set V of $\Gamma(Z_{3p})$ is $\{3, 6, 9, \dots, 3(p-1), p, 2p\}$. The total number of vertices in $\Gamma(Z_{3p})$ is $p + 1$. The vertex set of $\Gamma(Z_{3p})$ can be partitioned into two vertex subsets V_1 and V_2 where V_1 containing vertices of p multiples and V_2 containing vertices of 3 multiples. That is $V_1 = \{3, 6, 9, \dots, 3(p-1)\}$ and $V_2 = \{p, 2p\}$.

Let u and v be any two vertices in V_1 say $u = 6$ and $v = 9$. Then $uv = 6 \times 9 = 54$. Clearly $3p$ does not divide uv . Also let x and y be the two vertices in V_2 then $xy = p \times 2p = 2p^2$. Clearly $3p$ does not divide xy . Thus no two vertices in V_1 and V_2 are adjacent.

Also let us consider any one vertex in V_1 and any one in V_2 say $u=6$ and $v=2p$. Then $uv = 6 \times 2p = 12p$. clearly $3p$ divides $12p$. That is u and v are adjacent in $\Gamma(Z_{3p})$. In the same way, finally we get every vertex in V_1 are adjacent to all the vertices in V_2 and thus $\gamma(\Gamma(Z_{3p})) = 2$. Also these two vertices are both adjacent and having equal degree so both are connected and regular. Therefore $\gamma_{cr}(\Gamma(Z_{3p})) = \{ \text{Any one of the vertex in } V_1, \text{ any one of the vertex in } V_2 \} = 2$. Thus $\gamma_{cr}(\Gamma(Z_{3p})) = 2$. $\square$

Theorem 14. For any graph $\Gamma(Z_{5p})$, where $p > 5$ is any prime number then $\gamma_{cr}(\Gamma(Z_{5p})) = 2$.

Theorem 15. For any graph $\Gamma(Z_{7p})$, where $p > 7$ is any prime number then $\gamma_{cr}(\Gamma(Z_{7p})) = 2$.

Theorem 16. For any graph $\Gamma(Z_{pq})$, where p and $q > p$ are distinct prime numbers then $\gamma_{cr}(\Gamma(Z_{pq})) = 2$.

Proof. Using the above theorems, we get $\gamma_{cr}(\Gamma(Z_{3p})) = \gamma_{cr}(\Gamma(Z_{5p})) = \gamma_{cr}(\Gamma(Z_{7p})) = 2$. Similarly we get $\gamma_{cr}(\Gamma(Z_{11p})) = 2, p > 11$ and $\gamma_{cr}(\Gamma(Z_{13p})) = 2, p > 13$. Continue the same process, finally we get $\gamma_{cr}(\Gamma(Z_{pq})) = 2$. $\square$

Theorem 17. *If p and q are distinct primes where $p < q$ and n is a positive integer greater than one, then $\gamma_{cr}(\Gamma(Z_{p^n q})) = 2$.*

4 Connected Regular domination number of some other graphs in $\Gamma(Z_n)$

In this section we are calculating the connected regular domination number of $\Gamma(Z_n)$ for some $n = 4p, 6p, 8p, 2p^2, 3p^2, 2^n, 3^n, pqr$.

Theorem 18. *For any graph $\Gamma(Z_{4p})$, where $p > 5$ is any prime then $\gamma_{cr}(\Gamma(Z_{4p})) = 2$.*

Proof. The vertex set V of $\Gamma(Z_{4p})$ is $\{2, 4, 6, \dots, 2(2p-1), p, 2p, 3p\}$. The total number of vertices in $\Gamma(Z_{4p})$ is $2p+1$. The vertex set V can be partitioned into three vertex subsets namely V_1, V_2, V_3 where the vertex subset V_1 containing multiples of p , the vertex subset V_2 containing multiples of 4 and the vertex subset V_3 containing the multiples of 2 other than V_1 and V_2 . That is $V_1 = p, 2p, 3p$, $V_2 = 4, 8, \dots, 4(p-1)$ and $V_3 = 2, 6, 10, \dots, 2(2p-1)$, that is $|V_3(\Gamma(Z_{4p}))| = p-1$.

Let $u = 2p, v = p$ and $w = 3p$. Then we have $uv = 2pxp = 2p^2$, $uw = 2px3p = 6p^2$ and $vw = px3p = 3p^2$. Clearly $4p$ does not divide uv, uw and vw . That is u, v, w are non-adjacent vertices in $\Gamma(Z_{4p})$. Let $x = 8$ and $y = 4(p-1)$ be any two vertices in V_2 . Then $xy = 8(4(p-1)) = 32(p-1)$. Clearly $4p$ does not divide xy . So x and y are not adjacent in $\Gamma(Z_{4p})$. In the same way, we get that no two vertices of V_2 are adjacent in $\Gamma(Z_{4p})$. As V_3 contains only the pendent vertices, no two vertices of V_3 are adjacent in $\Gamma(Z_{4p})$.

Let $u = 2p$ be any vertex from V_1 and $v = 12$ be any vertex from V_2 . Then $uv = 2p(12) = 24p$. Clearly $4p$ divides $24p$. That is u and v are adjacent in $\Gamma(Z_{4p})$. In the same way we get every vertices in V_1 are adjacent to all the vertices in V_2 .

Let r be any vertex other than $2p$ in V_1 , say $r = p$ or $r = 3p$.

Let s be any vertex from V_3 say $s = 6$. Then $rs = p(6)$ or $3p(6) = 6p$ or $18p$. Clearly $4p$ does not divide rs . That is r and s are non-adjacent in $\Gamma(Z_{4p})$. So the vertex p and $3p$ are non-adjacent to any vertex of V_3 . Let $t = 2p$ be a vertex in V_1 . Then $st = 6(2p) = 12p$. Clearly, $4p$ divides $12p$. That is s and t are adjacent in $\Gamma(Z_{4p})$. Thus all the vertices in V_3 are only adjacent to the vertex $2p$ in V_1 .

Thus, the dominating set $D = \{2p, \text{any one of the vertex in } V_2\} = 2$. Hence $\gamma_{cr}(\Gamma(Z_{4p})) = 2$.

□

Theorem 19. For any graph $\Gamma(Z_{6p})$, where $p > 5$ is any prime number then $\gamma_{cr}(\Gamma(Z_{6p})) = 3$.

Theorem 20. For any graph $\Gamma(Z_{8p})$, where $p > 5$ is any prime number then $\gamma_{cr}(\Gamma(Z_{8p})) = 2$.

Theorem 21. For any graph $\Gamma(Z_{2p^2})$, where $p > 2$ is any prime number then $\gamma_{cr}(\Gamma(Z_{2p^2})) = 2$.

Proof. The vertex set V of $\Gamma(Z_{2p^2})$ is $\{2, 4, 6, \dots, 2(p^2 - 1), p, 2p, \dots, p(2p-1)\}$. The total number of vertices in $\Gamma(Z_{2p^2})$ is $p^2 + p - 1$. The vertex set V of $\Gamma(Z_{2p^2})$ can be partitioned into three vertex subsets namely V_1, V_2, V_3 where the vertex subset V_1 containing odd multiples of p , the vertex subset V_2 containing even multiples of p and the vertex subset V_3 containing multiples of 2 other than V_2 . Clearly V_3 contains only pendent vertices and $|V_3| = p(p-1)$.

The vertex p^2 of V_1 is adjacent to all the vertices of V_2 and V_3 . Thus the dominating set will contain the vertex p^2 . Clearly, V_3 contains only pendent vertices and it is clear that all the vertices of V_3 are only adjacent to the vertex p^2 of V_1 . Also the remaining vertices of V_1 other than p^2 is adjacent to all the vertices of V_2 , which forms a complete bipartite graph. Thus we have $\gamma(\Gamma(Z_{2p^2})) = \{ \text{the vertex } p^2, \text{any one of the vertex from } V_1 - \{p^2\} \text{ or any one of the vertex from } V_2 \}$.

Since the vertex p^2 is not adjacent to any vertex of V_1/p^2 and all the vertices of V_2 are adjacent to p^2 , the connected dominating set contains p^2 and any one of the vertex from V_2 . Also these connected dominating vertices have equal degree. We have $\gamma_{cr}(\Gamma(Z_{2p^2})) = \{ \text{the vertex } p^2, \text{any one of the vertex from } V_2 \}$. So $\gamma_{cr}(\Gamma(Z_{2p^2})) =$

2. □

Theorem 22. For any graph $\Gamma(Z_{3p^2})$, where $p > 3$ is any prime number then $\gamma_{cr}(\Gamma(Z_{3p^2})) = 2$.

Theorem 23. For any graph $\Gamma(Z_{2^n})$, where $n \geq 3$ is any positive integer then $\gamma_{cr}(\Gamma(Z_{2^n})) = 1$.

Proof. The vertex set V of $\Gamma(Z_{2^n})$ is $\{2, 4, 6, \dots, 2^{n-1}, 2(2^{n-1} - 1)\}$. The total number of vertices in $\Gamma(Z_{2^n})$ is $2^{n-1} - 1$. Since we know that the vertex 2^{n-1} is adjacent to all the vertices in $\Gamma(Z_{2^n})$. Thus $\gamma_{cr}(\Gamma(Z_{2^n})) = 1$. □

Theorem 24. For any graph $\Gamma(Z_{3^n})$, where $n > 3$ is any positive integer then $\gamma_{cr}(\Gamma(Z_{3^n})) = 1$.

5 Conclusions

In this paper, we evaluated the connected regular domination for complete, complete bipartite and some other graphs in $\Gamma(Z_n)$.

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