A Note on Partial Vertex Critical Domination of graphs

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Abstract
A graph $G$ is vertex domination critical if for any vertex $v$ of $G$, the domination number of $G - v$ is less than the domination number of $G$. A graph $G$ is partial vertex domination critical if it satisfies the following two conditions: $(i)$ $V = V_1 \cup V_2$, where $V_1$ and $V_2$ are vertex disjoint subsets of $V$, $(ii)$ there exists any vertex $v$ in $V_1$ or $V_2$, the domination number of $G - v$ is less than the domination number of $G$.

This paper analysis the different kind of vertex domination critical graphs in Brigham et al [1] and Fulman et al [4] papers. This review of literature also aims to encourag additional research on this topics, and concludes with several suggestions for further research.

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1 Introduction

An important consideration in the topological design of a network is fault tolerance, that is, the ability of the network to provide service even when it contains a faulty component or components. The behavior of a network in the presence of a fault can be analyzed
by determining the effect that removing an edge or a vertex from its underlying graph $G$ has on the fault tolerance criterion.

It is often of interest to know how the value of a graph parameter is affected when a small change is made in a graph, for instance vertex or edge removal and edge addition. In this connection, we consider the semi-expository paper by Carrington, Harary, and Haynes [3] that surveyed the problems of characterizing the graphs $G$ in the following six classes. Let $G - v$ (respectively, $G - e$) denote the graph formed by removing vertex $v$ (respectively edge $e$) from $G$. We use acronyms to denote the following classes of graphs (C represents changing; U-unchanging; V-vertex; E-edge; R-removal; A-addition.

(CVR) $\iff \gamma(G - v) \neq \gamma(G), \forall v \in V$

(CER) $\iff \gamma(G - e) \neq \gamma(G), \forall e \in E$

(CVA) $\iff \gamma(G + e) \neq \gamma(G), \forall e \in E(G)$

(UVR) $\iff \gamma(G - v) = \gamma(G), \forall v \in V$

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(UVA) $\iff \gamma(G + e) = \gamma(G), \forall e \in E(G)$

These six problems have been approached individually in the literature with other terminology and several related problems using the above "changing and unchanging" terminology first suggested by F. Harary [5]. If $\gamma(G - v) \neq \gamma(G)$ for all $v \in V$, then $\gamma(G - v) = \gamma(G) - 1$ for all $v \in V$, and so the graphs in CVR are precisely the vertex critical graphs introduced by Brigham et al [1].

We consider only finite simple graphs with vertex-set $V$ and edge-set $E$. A dominating set of a graph $G$ is a set $S$ of vertices of $G$ such that every vertex not in $S$ is adjacent to a vertex in $S$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum size of a dominating set. A set is independent (or stable) if no two vertices in it are adjacent. An independent dominating set of $G$ is a set that is both dominating and independent in $G$. The independent domination number of $G$, denoted by $i(G)$, is the minimum size of an independent dominating set. The independent number of $G$, denoted $\alpha(G)$, is the maximum size of an independent set in $G$. It follows immediately that $\gamma(G) \leq i(G) \leq \alpha(G)$.
2 Changing Vertex Domination of a Graph

Graphs whose domination number drops whenever an edge is added is called the edge critical graphs and graphs whose domination number drops whenever a vertex is deleted is called the vertex critical graphs. These two concepts are quite distinct, but have more in common than first meets in eye.

In this section, we concentrate our attention on vertex critical domination of graphs. Although we devote the great majority of this section to this property, we also touch on some of the work that has been done on other ways in which graphs may be extremely with respect to the domination number. A graph is \( k \)-domination vertex critical if \( \gamma(G) = k \) and for every vertex \( v \) of \( G \), \( \gamma(G - v) = k - 1 \). Vertex critical graphs are distinct from edge critical graphs. The cycle \( C_7 \) is vertex critical but not edge critical.

**Theorem 1.** [2] The only 1 - vertex critical graph is the graph on a single vertex. The 2 - vertex critical graphs are the graphs \( K_{2t} - F \), where \( F \) is a perfect matching.

Clearly they noted that every 2 - vertex critical graph is also a 2 - edge critical graph, but not conversely. We use the term domination critical to refer to graphs that are either vertex critical or edge critical. The following simple theorem is frequently useful.

**Theorem 2.** If \( G \) is any domination critical graph, then every vertex of \( G \) belongs to some minimum dominating set.

while the vertex critical and edge critical graphs are the main focus of this survey, there are two recently studied concepts that are closely related and deserve recognition here, too. There are many examples of \( k \) vertex critical graphs for \( k \geq 3 \), and several ways to generate them. Let \( A \) and \( B \) be two graphs with \( a \in A \) and \( b \in B \). The coalescence \( A.B \) of \( A \) and \( B \) via, \( a \) and \( b \), is the graph obtained from the disjoint union of \( A \) and \( B \) by identifying the vertex \( a \) and \( b \). Brigham, Chinn and Dutton [1] discuss this operation and prove the following result.

**Theorem 3.** [1] A coalescence of two graphs \( A \) and \( B \) is vertex critical if and only if both of \( A \) and \( B \) are vertex critical.
Theorem 4. [2] If $H$ is a coalescence of $A$ and $B$, then $\gamma(A) + \gamma(B) \leq \gamma(H) \leq \gamma(A) + \gamma(B)$.

Theorem 5. Let $G$ be a connected, vertex critical graph with order $n$. Then i) [1] $n \leq (k - 1)(\Delta + 1) + 1$, and ii) [4] if $G$ has exactly $(k - 1)(\Delta + 1) + 1$ vertices, then $G$ is regular.

In fact, [1] shows that the bound in part (i) of the previous theorem holds for any graph that contains at least one critical vertex. Vertex critical graphs must always be point distinguishing as observed by Brigham et al [1]. They note that vertex critical graphs do not contain a pair of vertices $u$ and $v$ with $N[v] \subseteq N[u]$. Brigham et al [1] show that vertex critical graphs with order $n$ and size $m$ satisfy the following relation between $m$, $\gamma$, $\Delta$ and $n$.

A graph $G$ is vertex domination-critical or $\gamma$-critical or simply critical, if any vertex $v$ of $G$, $\gamma(G - v) < \gamma(G)$. In [6], Brigham et al, showed that a forbidden subgraph characterization of graphs in CVR is not possible. However, they characterized those graphs in CVR that have the minimum order of vertex critical graphs $n = \gamma(G) + \Delta(G)$. In [1], Brigham et al studied $\gamma$-critical graphs and posed the following questions:

1) If $G$ is a $\gamma$-critical graph, is $|V| \geq (\delta + 1)(\gamma - 1) + 1$? 2) If a $\gamma$-critical graph $G$ has $(\Delta + 1)(\gamma - 1) + 1$ vertices, is $G$ regular? 3) Does $i = \gamma$ for all $\gamma$-critical graphs? 4) Let $d$ be the diameter of the $\gamma$-critical graph $G$. Does $d \leq 2(\gamma - 1)$ always hold?

In [4], Fulman, Hanson, and MacGillivray, consider the circulant graph $G = X(17; 1, 3, 5, 7, 10, 12, 14, 16)$, i.e., the graph with vertex-set $\{0, 1, \ldots, 16\}$ in which vertices $i$ and $j$ are adjacent, if and only if, $i - j$ is congruent to one of 1, 3, 5, 7, 10, 12, 14, or 16 modulo 17. This graph is 8-regular, 3-critical, and $|V| = 17 < (8 + 1)(3 - 1) + 1 = 19$, which gives a negative answer for question one. Further, it can be readily verified that the circulant $G$ is 3-critical, and $i(G) = 5 > \gamma(G) = 3$, which gives a negative answer for the third question. Similarly, they proved the second and fourth questions as follows:

Theorem 6. If $G$ is $\gamma$-critical with $|V| = (\Delta + 1)(\gamma - 1) + 1$, then $G$ is regular. [4]

Theorem 7. The diameter of $d$ of a $\gamma$-critical graph $G$ satisfies $d \leq 2(\gamma - 1)$ for $\gamma \geq 2$. [4]
Finally, J. Fulman et al, show that the first and third questions have negative answers and the others have positive answers.

3 Main Results

In this section, we show that the second question has a negative answer and the others have positive answers. Before that we defined the partially vertex critical domination set. A graph $G$ is partial vertex domination critical if it satisfies the following two conditions:

1. $V = V_1 \cup V_2$, where $V_1$ and $V_2$ are vertex disjoint subsets of $V$,
2. there exists any vertex $v$ in $V_1$ or $V_2$, the domination number of $G - v$ is less than the domination number of $G$.

**Theorem 8.** For any graph $K_{2,n}$ where $n > 2$, and $K_{2,n} \in CVR$ has order $|V| = (\Delta + 1)(\gamma - 1) + 1$, then $K_{2,n}$ is non-regular.

**Proof.** Let $V_1$ and $V_2$ be the two vertex subsets of $V$, where $|V_1| = 2$, $|V_2| = n$. Since, we know that there exists almost two vertices $u$ and $v$ in $K_{2,n}$, they have the maximum degree $n$. That is, $\text{deg}(v) = \text{deg}(u) = \Delta = n$. The number of vertices in $K_{2,n}$ is $n + 2$. Clearly, there exists any other vertex $w$ which is not equal to $u$ and $v$ in $K_{2,n}$ such that $w$ is adjacent to both $u$ and $v$.

Let $D$ be the domination set of $K_{2,n}$. Clearly, $D = \{u, v\}$, and $\gamma(K_{2,n}) = |D| = 2$, then, $(\Delta + 1)(\gamma - 1) + 1 = (n + 1)(2 - 1) + 1 = n + 2 = |V(K_{2,n})|$. Since, $\text{deg}(u) = \text{deg}(v) \neq \text{deg}(w)$ and hence $K_{2,n}$ is non-regular. 

For example, in $K_{2,4}$, only two vertices have degree 4 and remaining vertices have degree 2. Thus, $K_{2,4}$ is non-regular graph.

**Theorem 9.** For any graph $K_{2,n}$ where $n \geq 2$, is a $\gamma$-critical, then $|V| \geq (\delta + 1)(\gamma - 1) + 1$.

**Proof.** Using theorem (3.1), we know that the minimum degree of $K_{2,n}$ is 2 and the domination number of $K_{2,n}$ is 2. Then, $|V| \geq (2 + 1)(2 - 1) + 1 = 3 + 1 = 4$. Since, $|V| = n + 2$, where $n \geq 2$. Clearly, $|V| \geq 4$. Hence, it is proved.

For example, in $K_{2,6}$, $\gamma = 2$ and $\delta = 2$, then $|V(K_{2,6})| = 8 \geq (\delta + 1)(\gamma - 1) + 1 = (2 + 1)(2 - 1) + 1 = 4$. Similarly, in $K_{2,n}$
where \( n \geq 2 \), then \( i = \gamma \) for all 2-critical graphs and if \( d \) be the
diameter of the 2-critical graph \( K_{2,n} \) then \( d \leq 2(\gamma - 1) \) always holds.

Finally, if we use partially vertex critical domination conditions,
it may be concluded from Brigham et al [1] two questions given need
not be sufficient.

References


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