

PRIVATE NEIGHBOURHOOD AND ENRESDOWED GRAPHS

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Abstract

Let $G=(V,E)$ be a simple, finite, non-trivial and undirected graph. The Restrained dominating set of a graph is a dominating set in which every vertex in $V - D$ is adjacent to some vertex in $V - D$ [10]. $k - \gamma_r$ enresdowed graph is one in which every restrained dominating set of, cardinality k contains a minimum restrained dominating set. In this paper, Some bounds were calculated for $k - \gamma_r$ enresdowed graph related with its private neighbourhood.

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1 Introduction

The definition of domination, restrained domination and private neighbourhood are as in [10]. The ordinary restrained dominating set is denoted as rd set and the minimum rd set is denoted as γ_r set. A rd set of G with cardinality k which contains a γ_r set of G is called $k - \gamma_r$ enresdowed graph. The n - friendship graph is a graph which is obtained by attaching friendship graph $f_1, s_i, s_i \geq 2$ to every vertex of a path with n vertices.

Let k be a positive integer. A simple graph $G=(V,E)$ is called a $k - \gamma_r$ enresdowed graph if every rd set of G of cardinality k contains a γ_r set of G .

Let P_n be a path of n vertices namely, $u_1, u_2, \dots, u_i \dots, u_n$. To the vertex $u_i, 1 \leq i \leq n$ attach a friendship graph $f_1, s_i, s_i \geq 2$. The graph obtained is called n - friendship graph and it is denoted by $f_{s_1, s_2, \dots, s_i \dots, s_n}$. The vertices $u_1, u_2, \dots, u_i \dots, u_n$ are called nodes of the n - friendship graph.

Corollary 1. *The n - friendship graph has a unique γ_r set.*

2 Enresdowedness of graphs and private neighbours

Theorem 2. *G be a n - friendship graph with nodes $u_1, u_2, \dots, u_i \dots, u_n$ then every node has atleast four private neighbours in $V - D$.*

Proof. Let $D = \{u_1, u_2, \dots, u_i \dots, u_n\}$ be the set of nodes. Let $v_1, w_1, v_2, w_2, \dots, v_s, w_s$ be the vertices adjacent to some $u_i, 1 \leq i \leq n$ and the vertices v_j, w_j are adjacent for $1 \leq j \leq s$.

Suppose $u_i, 1 \leq i \leq n$ has no private neighbours then the neighbours of u_i say v_j or $w_j, 1 \leq j \leq s$ is adjacent to some $u_r, 1 \leq r \leq n, r \neq i$ which is a contradiction.

Suppose $u_i, 1 \leq i \leq n$ has exactly one private neighbour then u_i is adjacent to a pendant vertex or u_i is attached to vertex v_j ,

$1 \leq j \leq s$ say and $w_j, 1 \leq j \leq s$ is adjacent to some other $u_r, 1 \leq r \leq n, i \neq r$ which is a contradiction.

Suppose $u_i, 1 \leq i \leq n$ has two private neighbours then either u_i is attached to a vertices v_j and $w_j, 1 \leq j \leq s$ and where the vertices v_{j+1} and $w_{j+1}, 1 \leq j \leq s$ are adjacent to some other $u_r, 1 \leq r \leq n, i \neq r$ which is a contradiction or u_i is attached to the vertices v_j and $v_{j+1}, 1 \leq j \leq s$, where v_j and v_{j+1} are non-adjacent vertices then the vertices w_j and $w_{j+1}, 1 \leq j \leq s$ are adjacent to some other $u_r, 1 \leq r \leq n, r \neq i$ which is a contradiction. Suppose if u_i is attached to a vertices v_j and $w_{j+1}, 1 \leq j \leq s$, where v_j and w_{j+1} are non-adjacent vertices then the vertices w_j and $v_{j+1}, 1 \leq j \leq s$ are adjacent to some other $u_r, 1 \leq r \leq n, i \neq r$ which is a contradiction or if u_i is attached to a vertices w_j and $v_{j+1}, 1 \leq j \leq s$, where w_j and v_{j+1} are non-adjacent vertices then the vertices v_j and $w_{j+1}, 1 \leq j \leq s$ are adjacent to some other $u_r, 1 \leq r \leq n, r \neq i$, which is a contradiction. Similarly if u_i is attached to a vertices w_j and $w_{j+1}, 1 \leq j \leq s$ where w_j and w_{j+1} are non-adjacent vertices then the vertices v_j and $v_{j+1}, 1 \leq j \leq s$ are adjacent to some other $u_r, 1 \leq r \leq n, i \neq r$ which is a contradiction.

Suppose if u_i is attached to a vertices v_{j+1} and $w_{j+1}, 1 \leq j \leq s$ where v_{j+1} and w_{j+1} are adjacent vertices then the vertices v_j and $w_j, 1 \leq j \leq s$ are adjacent to some other $u_r, 1 \leq r \leq n, r \neq i$ which is a contradiction.

Suppose u_i has three private neighbours then either u_i is attached to a vertices v_j, w_j and $v_{j+1}, 1 \leq j \leq s$, and where the vertex $w_{j+1}, 1 \leq j \leq s$ are adjacent to some other $u_r, 1 \leq r \leq n, i \neq r$ which is a contradiction or u_i is attached to a vertices v_j, w_j and $w_{j+1}, 1 \leq j \leq s$, and where the vertex $v_{j+1}, 1 \leq j \leq s$ are adjacent to some other $u_r, 1 \leq r \leq n, r \neq i$, which is a contradiction. Similarly if u_i is attached to a vertices, w_j, v_{j+1} and $w_{j+1}, 1 \leq j \leq s$, and where the vertex $v_j, 1 \leq j \leq s$ are adjacent to some other $u_r, 1 \leq r \leq n, i \neq r$ which is a contradiction or u_i is attached to a vertices, v_j, v_{j+1} and $w_{j+1}, 1 \leq j \leq s$ and where the vertex $w_j, 1 \leq j \leq s$ are adjacent to some other $u_r, 1 \leq r \leq n, r \neq i$ which is a contradiction. Hence the theorem.

□

Theorem 3. Let G be a n – friendship graph with a set of nodes $\{u_1, u_2, \dots, u_s, \dots, u_n\}$. Let $\eta = \min_{1 \leq s \leq n} \{|pn(u_s, D)|\}$. Then there exist an t , $1 \leq t \leq n$ such that $pn(u_t, D) = \eta$, $(D - \{u_t\}) \cup (pn(u_t, D) - \bigcup_{j=1}^l \{v_j\})$, where v_j belongs to $pn(u_t, D)$ for $1 \leq j \leq l$, $l \geq 2$, is a rd set of G of cardinality $k + \eta/2 - 1$ or $(D - \{u_t\}) \cup (pn(u_t, D) - \bigcup_{j=1}^l \{w_j\})$ where w_j belongs to $pn(u_t, D)$ for $1 \leq j \leq l$, $l \geq 2$, is a rd set of G of cardinality $k + \eta/2 - 1$.

Proof. Let $D = \{u_1, u_2, \dots, u_s, \dots, u_n\}$ be a unique γ_r set of a n – friendship graph G . Let the $\langle pn(u_s, D) \rangle$ $1 \leq s \leq k$ is a union of K'_2 s. Given $\eta = \min_{1 \leq s \leq n} \{|pn(u_s, D)|\}$, then there exist an t , $1 \leq t \leq k$ such that $pn(u_t, D) = \eta$. Let the vertex set of $pn(u_t, D)$ be $\{v_1, w_1, v_2, w_2, \dots, v_l, w_l\}$, $l \geq 2$ which is adjacent to some u_s , $1 \leq s \leq n$ and the vertices v_j, w_j are adjacent for $1 \leq j \leq l$. Consider the sets D_1, D_2 and D_3 in such a way,
 $D_1 = \{u_1, u_2, \dots, u_{t-1}, u_{t+1}, \dots, u_n\}$ of cardinality $k - 1$,
 $D_2 = \bigcup_{j=1}^l \{v_j\}$ of cardinality l and $D_3 = \bigcup_{j=1}^l \{w_j\}$,
of cardinality l . For $l \geq 2$, where v_j, w_j belongs to $pn(u_t, D)$. Hence $D_1 = D - \{u_t\}$.

Here $pn(u_t, D) = \{v_1, w_1, v_2, w_2, \dots, v_l, w_l\}$
 $pn(u_t, D) = \{v_1, v_2, \dots, v_l\} \cup \{w_1, w_2, \dots, w_l\}$
 $pn(u_t, D) = \left(\bigcup_{j=1}^l \{v_j\}\right) \cup \left(\bigcup_{j=1}^l \{w_j\}\right)$
 $pn(u_t, D) = D_2 \cup D_3$.
Therefore $pn(u_t, D) - \left(\bigcup_{j=1}^l \{v_j\}\right) = pn(u_t, D) - D_2$ and
 $pn(u_t, D) - \left(\bigcup_{j=1}^l \{w_j\}\right) = pn(u_t, D) - D_3$.
Let $D_4 = D_1 \cup D_2$ and $D_5 = D_1 \cup D_3$.

Part (i) Consider a set $(D - \{u_t\}) \cup (pn(u_t, D) - \bigcup_{j=1}^l \{v_j\})$
 $l \geq 2$. This set is nothing but D_4 . In this set D_4 , the vertices $\{u_1, u_2, \dots, u_{t-1}, u_{t+1}, \dots, u_n\}$ dominates its corresponding private neighbours v_j or w_j and the vertices v_j, w_j are adjacent for $1 \leq j \leq l$ and the set $\{v_1, v_2, \dots, v_l\}$ dominates $\{w_1, w_2, \dots, w_l\}$ and u_t for $1 \leq t \leq n$, $l \geq 2$. Then each vertex w_j $1 \leq j \leq l$ is adjacent to u_t in $V - D$ and u_t is adjacent to the vertices u_{t-1} and u_{t+1} in D_1 . Thus D_4 is a rd set and the cardinality of the set D_4 is the sum of the cardinality $k - 1$ of the set D_1 and the cardinality $\eta/2$ of the set

$\{v_1, v_2, \dots, v_l\}$. Thus the cardinality of the set D_4 is $k + \eta/2 - 1$.

Part (ii) Consider a set $(D - \{u_t\}) \cup (\text{pn}(u_t, D) - \bigcup_{j=1}^l \{w_j\})$ $l \geq 2$. This set is nothing but D_5 . In this set D_5 , the vertices $\{u_1, u_2, \dots, u_{t-1}, u_{t+1}, \dots, u_n\}$ dominates its corresponding private neighbours v_j or w_j and the vertices v_j, w_j are adjacent for $1 \leq j \leq l$ and the set $\{w_1, w_2, \dots, w_l\}$ dominates $\{v_1, v_2, \dots, v_l\}$ and u_t for $1 \leq t \leq n, l \geq 2$. Then each vertex $v_j, 1 \leq j \leq l$ is adjacent to u_t in $V - D$ and u_t is adjacent to the vertices u_{t-1} and u_{t+1} in D_1 . Thus D_5 is a rd set and the cardinality of the set D_5 is the sum of the cardinality $k - 1$ of the set D_1 and the cardinality $\eta/2$ of the set $\{w_1, w_2, \dots, w_l\}$. Thus the cardinality of the set D_5 is $k + \eta/2 - 1$.

Thus $(D - \{u_t\}) \cup (\text{pn}(u_t, D) - \bigcup_{j=1}^l \{v_j\})$, or $(D - \{u_t\}) \cup (\text{pn}(u_t, D) - \bigcup_{j=1}^l \{w_j\})$, where v_j or w_j belongs to $\text{pn}(u_t, D)$ for $1 \leq j \leq l, l \geq 2$, is a rd set of G of cardinality $k + \eta/2 - 1$. $\square$

Theorem 4. Let G be a n - friendship graph $f_{s_1, s_2, \dots, s_i, \dots, s_n}$ with unique γ_r set $D = \{u_i\}_{i=1}^n, n \geq 2$ and $\langle \text{pn}(u_i, D) \rangle, 1 \leq i \leq n$ be a union of lK'_2 s, $l \geq 2$ with s_i vertices where $s_i \neq s_j$, for $i \neq j, 1 \leq i, j \leq n, s_i, s_j \geq 4$ and atleast one $s_i = 4$. Then G is $\gamma_r - \gamma_r$ enresdowed but not $m - \gamma_r$ enresdowed for any m ,

$$\gamma_r + 1 \leq m \leq \sum_{i \neq m}^n s_i/2 + (\gamma_r - 1).$$

Proof. Let G be a n - friendship graph $f_{s_1, s_2, \dots, s_i, \dots, s_n}$ with unique γ_r set, $D = \{u_1, u_2, \dots, u_i, \dots, u_n\}, n \geq 2$. Let $\langle \text{pn}(u_i, D) \rangle, 1 \leq i \leq n$ is a union of lK'_2 s, $l \geq 2$ with s_i vertices, where $s_i \neq s_j$, for $i \neq j, 1 \leq i, j \leq n, s_i, s_j \geq 4$. Let $v_1, w_1, v_2, w_2, \dots, v_r, w_r, \dots, v_{s_i}, w_{s_i}, s_i \geq 4$ be the set of vertices adjacent to some $u_i, 1 \leq i \leq m$ and the vertices $v_r, w_r, 1 \leq r \leq s$ are adjacent.

For the vertex u_1 , let $\langle \text{pn}(u_1, D) \rangle$ be a union of lK_2 s, $l \geq 2$ with the vertex set $\{v_1, w_1, v_2, w_2, \dots, v_{s_1}, w_{s_1}\}, s_1 \geq 4$. In general for any vertex u_i , the $\langle \text{pn}(u_i, D) \rangle, 1 \leq i \leq n$ be the union of lK_2 s with the vertex set $\{v_{s_{i-1}+1}, w_{s_{i-1}+1}, v_{s_{i-1}+2}, w_{s_{i-1}+2}, \dots, v_{s_i}, w_{s_i}\}, s_i \geq 4$. The vertex set of G is $\{u_1, u_2, \dots, u_i, \dots, u_n, v_1, w_1, v_2, w_2, \dots, v_{s_1}, w_{s_1}, \dots, v_{s_{i-1}+1}, w_{s_{i-1}+1}, v_{s_{i-1}+2}, w_{s_{i-1}+2}, \dots, v_{s_i}, w_{s_i}, \dots, v_{s_{n-1}+1}, w_{s_{n-1}+1}, v_{s_{n-1}+2}, w_{s_{n-1}+2}, \dots, v_{s_n}, w_{s_n}\}$ where each s_i vertices are dominated by $u_i, 1 \leq i \leq n$. Since $D = \{u_1, u_2, \dots, u_i, \dots, u_n\}$ is the

γ_r set of G . Clearly G is $\gamma_r - \gamma_r$ enresdowed.

Consider any sets with cardinality m , where $m > \gamma_r + 1$, the $\langle \text{pn}(u_i, D) \rangle$ for $1 \leq i \leq n$ contains lK'_2 s, $l \geq 2$ then there exist two cases $l = 2i$ and $l = 2i+1$, $i \geq 1$.

Case (i) Suppose $l = 2i$, $i \geq 1$.

Then take $\langle lK_2 \rangle = Q(K_2)$. Let the vertex set of $Q(K_2)$ be

$$\{v_{s_{2i-1}+1}, w_{s_{2i-1}+1}, v_{s_{2i-1}+2}, w_{s_{2i-1}+2}, \dots, v_{s_{2i}}, w_{s_{2i}}, \dots\}, s_{2i} \geq 4.$$

For the $\langle \text{pn}(u_i, D) \rangle$ of the type $Q(K_2)$, consider either a set

$$D_{2i} = \{u_1, u_2, \dots, u_{2i-1}, u_{2i+1}, \dots, u_{n-1}\} \cup \{v_{s_{2i-1}+1}, v_{s_{2i-1}+2}, \dots, v_{s_{2i}}\}$$

$$\text{or } D_{2i} = \{u_1, u_2, \dots, u_{2i-1}, u_{2i+1}, \dots, u_{n-1}\} \cup \{w_{s_{2i-1}+1}, w_{s_{2i-1}+2}, \dots, w_{s_{2i}}\}.$$

Therefore D_{2i} is of cardinality $(s_{2i}/2 + n - 1)$.

Case(ii) Suppose $l = 2i+1$, $i \geq 1$. Then take $\langle lK_2 \rangle = R(K_2)$.

Let the vertex set of $R(K_2)$ be

$$\{v_{s_{(2i+1)-1}+1}, w_{s_{(2i+1)-1}+1}, v_{s_{(2i+1)-1}+2}, w_{s_{(2i+1)-1}+2}, \dots, v_{s_{(2i+1)}}, w_{s_{(2i+1)}}\}$$

$s_{2i+1} \geq 4$ otherwise,

$$R(K_2) = \{v_{s_{2i}+1}, w_{s_{2i}+1}, v_{s_{2i}+2}, w_{s_{2i}+2}, \dots, v_{s_{(2i+1)}}, w_{s_{(2i+1)}}\}$$

For the $\langle \text{pn}(u_i, D) \rangle$ of the type $R(k_2)$, consider either a set

$$D_{2i+1} = \{u_1, u_2, \dots, u_{2i}, u_{2i+2}, \dots, u_{n-1}\} \cup \{v_{s_{2i}+1}, v_{s_{2i}+2}, \dots, v_{s_{(2i+1)}}\} \text{ or}$$

$$D_{2i+1} = \{u_1, u_2, \dots, u_{2i}, u_{2i+2}, \dots, u_{n-1}\} \cup \{w_{s_{2i}+1}, w_{s_{2i}+2}, \dots, w_{s_{(2i+1)}}\}.$$

Therefore D_{2i+1} is of cardinality $(s_{2i+1}/2 + n - 1)$.

$$\text{Let } D_n = \{u_1, u_2, \dots, u_{n-1}\} \cup \{v_{s_{n-1}+1}, v_{s_{n-1}+2}, \dots, v_{s_n}\}.$$

D_n is of cardinality $(s_n/2 + n - 1)$. Let $D_2 = (\bigcup_{\substack{i \geq 1 \\ 2i \neq n}} D_{2i}) \cup D_n$ and

$$D_3 = (\bigcup_{\substack{i \geq 1 \\ 2i+1 \neq n}} D_{2i+1}) \cup D_n, \text{ Take } D_4 = (\bigcup_{\substack{i \geq 1 \\ 2i \neq n}} D_{2i}) \cup (\bigcup_{\substack{i \geq 1 \\ 2i+1 \neq n}} D_{2i+1}) \cup$$

$$D_n. D_4 \text{ is a rd set of cardinality } \sum_{\substack{i=1 \\ i \neq m}}^n s_i/2 + (\gamma_r - 1), \text{ which does not}$$

contain the u_n . Hence D_4 is a rd set not containing D . Hence G is not $m - \gamma_r$ enresdowed for any m , $\gamma_r + 1 \leq m \leq \sum_{\substack{i=1 \\ i \neq m}}^n s_i/2 + (\gamma_r - 1)$.

□

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References

- [1] Anunchuen.N and Caccetta.L, On Minimally k - Extendable Graphs, School of Mathematics and Statistics , Curtin University of Technology, GPO Box U1987, Perth 6001.
- [2] Anunchuen.N and Caccetta.L, On Critically k - Extendable Graphs, Australasian Journal Of Combinatorics ,6(1992), 39 - 65.
- [3] Arumugam.S, Sivagnanam.C, Neighborhood Total Domination in Graphs, OPUSCULE Mathematica Vol. 31. No .4. 2011.
- [4] Berge.C, Theory of graphs and its applications, Dunod, Paris, 1958.
- [5] Bondy.J.A and Murty U.S.R, Graph Theory with Applications, The MacMillan Press, London, (1976).
- [6] Chartrand, L . Lesniak, Graphs and Digraphs, Second ed., Wadsworth and Brooks/Cole, Monterey CA, 1986
- [7] Cockayne.E and Hedetniemi.S, Towards a theory of domination in graphs, Networks Fall(1977) ,247 - 261.
- [8] Domke.G.S, Hattingh.J.H Hedetniemi.S.T, Laskar.R.C., Markus .L.R, Restrained domination in graphs, Discrete Math.203(1999), 61 - 69.
- [9] Harary.F, Graph Theory, Addison Wesley, Reading Mass(1972).
- [10] Teresa W.Haynes, Stephen .Hedetniemi, Peter J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, INC. New York, 1998.

