QUOTIENT-3 CORDIAL LABELING FOR PATH RELATED GRAPHS-PART-I

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Abstract

In this paper, a new type of labeling namely quotient-3 cordial labeling \( f \) is introduced. Let \( G(V,E) \) be a simple graph of order \( p \) and size \( q \). Let \( f : V(G) \rightarrow \mathbb{Z}_4 - \{0\} \) be a function. For each edge \( E(G) \) define the labeling \( f^* : E(G) \rightarrow \mathbb{Z}_3 \) by \( f^*(uv) = \left\lceil \frac{f(u)}{f(v)} \right\rceil \mod 3 \) where \( f(u) \geq f(v) \). The function \( f \) is called quotient-3 cordial labeling of \( G \) if the number of vertices labeled with \( i \) and the number of vertices labeled with \( j \) differ by at most 1, the number of edges labeled with \( k \) and the number of edges labeled with \( l \) differ by at most 1. \( 1 \leq i, j \leq 3, i \neq j \) and \( 0 \leq k, l \leq 2, k \neq l \).
Here we proved that \( P_n \) and some path related graphs like \( Y \)-tree, \((P_n; S_2)\) and \((P_n; C_3)\) are quotient-3 cordial graphs.

**AMS Subject Classification:** 05C78

**Key Words and Phrases:** Path, \( Y \)-tree, quotient-3 cordial graph.

1 Introduction

Here the graphs considered are finite, simple, undirected and non trivial. Graph theory has a good development in the graph labeling and has a broad range of applications. Refer Gallian [1] for more information. The cordial labeling concept was first introduced by Cahit [2]. \( H\)- and \( H_2\)-cordial labeling was introduced by Freeda S and Chellathurai R. S [3]. Mean Cordial Labeling was introduced by Albert William, Indra Rajasingh, and Roy S [4]. 3-product cordial labeling was introduced by Jeyanthi P and Maheswari A [5]. Quotient-3 cordial labeling was introduced by Sumathi P, Mahalakshmi A and Rathi A [6]. we follow [7] for notations and terminology. A graph \( G \) is said to be quotient-3 cordial graph if it receives quotient-3 cordial labeling. Let \( v_f(i) \) denotes the number of vertices labeled with \( i \) and \( e_f(k) \) denotes the number of edges labeled with \( k \), \( 1 \leq i \leq 3 \), \( 0 \leq k \leq 2 \).

2 Definitions

**Definition 1.** A \( Y \)-tree is a graph obtained from the path \( P_n \) by joining an edge to a vertex of the path \( P_n \) adjacent to an end vertex.

**Definition 2.** \((P_n; S_2)\) is a graph obtained by attaching the root of a star \( S_2 \) at each vertex of a path \( P_n \) through an edge.

**Definition 3.** A graph which is obtained from a path \( P_n \) by joining the vertex of a cycle \( C_3 \) to every vertex of a path through an edge is denoted by \((P_n; C_3)\)
3 Main Result

Definition 4. Let $G(V, E)$ be a simple graph of order $p$ and size $q$. Let $f : V(G) \to \mathbb{Z}_4 - \{0\}$ be a function. For each edge set $E(G)$ define the labeling $f^* : E(G) \to \mathbb{Z}_3$ by $f^*(uv) = \left\lceil \frac{f(u)}{f(v)} \right\rceil \pmod{3}$ where $f(u) \geq f(v)$. The function $f$ is called quotient-3 cordial labeling of $G$ if the number of vertices labeled with $i$ and the number of vertices labeled with $j$ differ by at most 1, the number of edges labeled with $k$ and the number of edges labeled with $l$ differ by at most 1. $1 \leq i, j \leq 3$, $i \neq j$ and $0 \leq k, l \leq 2$, $k \neq l$.

Illustration 5. A quotient-3 cordial graph

![Figure 1:]

Theorem 6. All paths $P_n$ are quotient-3 cordial graph.

Proof. Let $V(G) = \{u_i : 1 \leq i \leq n\}$ and $E(G) = \{(u_iu_{i+1}) : 1 \leq i \leq n - 1\}$
Here $|V(G)| = n$, $|E(G)| = n - 1$.
Define $f : V(G) \to \mathbb{Z}_4 - \{0\}$

Case (i): When $n \equiv 0, 1, 4, 5 \pmod{6}$
For $i$, $1 \leq i \leq n$
$f(u_i) = 1$, if $i \equiv 1, 2 \pmod{6}$
$f(u_i) = 3$, if $i \equiv 0, 3 \pmod{6}$
$f(u_i) = 2$, if $i \equiv 4, 5 \pmod{6}$

Case (ii): When $n \equiv 2 \pmod{6}$
Labeling of $u_i$, $1 \leq i \leq n - 1$ are similar to case (i).
Assign $f(u_n) = 3$. 
Case (iii): When $n \equiv 3 \pmod{6}$
Labeling of $u_i, 1 \leq i \leq n - 2$ are similar to case (i).
Assign $f(u_{n-1}) = 3, f(u_n) = 2$.
Table 1 gives the quotient-3 cordial for path $P_n$

<table>
<thead>
<tr>
<th>Nature of $n$</th>
<th>$v_f(1)$</th>
<th>$v_f(2)$</th>
<th>$v_f(3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \equiv 0, 3 \pmod{6}$</td>
<td>$\frac{n}{3}$</td>
<td>$\frac{n}{3}$</td>
<td>$\frac{n}{3}$</td>
</tr>
<tr>
<td>$n \equiv 1, 4 \pmod{6}$</td>
<td>$(\frac{n-1}{3}) + 1$</td>
<td>$(\frac{n-1}{3})$</td>
<td>$(\frac{n-1}{3})$</td>
</tr>
<tr>
<td>$n \equiv 2 \pmod{6}$</td>
<td>$(\frac{n+1}{3})$</td>
<td>$(\frac{n+1}{3}) - 1$</td>
<td>$(\frac{n+1}{3}) - 1$</td>
</tr>
<tr>
<td>$n \equiv 5 \pmod{6}$</td>
<td>$(\frac{n-2}{3}) + 1$</td>
<td>$(\frac{n-2}{3}) + 1$</td>
<td>$(\frac{n-2}{3}) + 1$</td>
</tr>
</tbody>
</table>

Table 1:

<table>
<thead>
<tr>
<th>Nature of $n$</th>
<th>$e_f(0)$</th>
<th>$e_f(1)$</th>
<th>$e_f(2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \equiv 0 \pmod{6}$</td>
<td>$\frac{n}{3} - 1$</td>
<td>$\frac{n}{3}$</td>
<td>$\frac{n}{3}$</td>
</tr>
<tr>
<td>$n \equiv 1, 4 \pmod{6}$</td>
<td>$(\frac{n-1}{3})$</td>
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<tr>
<td>$n \equiv 3 \pmod{6}$</td>
<td>$\frac{n}{3}$</td>
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<td>$\frac{n}{3}$</td>
</tr>
<tr>
<td>$n \equiv 5 \pmod{6}$</td>
<td>$(\frac{n-2}{3}) + 1$</td>
<td>$(\frac{n-2}{3}) + 1$</td>
<td>$(\frac{n-2}{3}) + 1$</td>
</tr>
</tbody>
</table>

Theorem 7. The $Y$-tree is quotient-3 cordial for $n > 2$.

Proof. Let $V(G) = \{u, u_i : 1 \leq i \leq n \}$ and $E(G) = \{(u_i u_{i+1}), (uu_{n-1}) : 1 \leq i \leq n - 1\}$
Here $|V(G)| = n + 1, |E(G)| = n$.
Define $f : V(G) \to \mathbb{Z}_4 - \{0\}$
When $n = 3$, define $f(u_1) = f(u_2) = 1, f(u_3) = 3$ and $f(u) = 2$.
Clearly this gives the quotient-3 cordial.
When $n > 3$
In this, when $n \equiv 1 \pmod{6}, n \equiv 2 \pmod{6}, n \equiv 3 \pmod{6}, n \equiv 4 \pmod{6}$ and $n \equiv 5 \pmod{6}$
Labeling of $u_i, 1 \leq i \leq n$ are similar to the respective cases as in theorem 6
When $n \equiv 0 \pmod{6}$
For $i, 1 \leq i \leq n$,$f(u_1) = 1$
$f(u_i) = 3, \text{if } i \equiv 2, 5 \pmod{6}$
\( f(u_i) = 2, \) if \( i \equiv 3, 4 \pmod{6} \)
\( f(u_i) = 1, \) if \( i \equiv 0, 1 \pmod{6} \)
Labeling of the vertex \( u \) is given below.
When \( n \equiv 0, 1, 3, 5 \pmod{6}, f(u) = 3 \)
When \( n \equiv 2, 4 \pmod{6}, f(u) = 2. \)
Table 2 gives the quotient-3 cordial graph for \( Y \)-tree for \( n > 3 \)

Table 2:

<table>
<thead>
<tr>
<th>Nature of ( n )</th>
<th>( v_f(1) )</th>
<th>( v_f(2) )</th>
<th>( v_f(3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n \equiv 0, 3 \pmod{6} )</td>
<td>( \frac{n}{3} )</td>
<td>( \frac{n}{3} )</td>
<td>( \frac{n}{3} + 1 )</td>
</tr>
<tr>
<td>( n \equiv 1 \pmod{6} )</td>
<td>( \frac{n+2}{3} )</td>
<td>( \frac{n+2}{3} - 1 )</td>
<td>( \frac{n+2}{3} )</td>
</tr>
<tr>
<td>( n \equiv 2, 5 \pmod{6} )</td>
<td>( \frac{n+1}{3} )</td>
<td>( \frac{n+1}{3} - 1 )</td>
<td>( \frac{n+1}{3} )</td>
</tr>
<tr>
<td>( n \equiv 4 \pmod{6} )</td>
<td>( \frac{n-1}{3} + 1 )</td>
<td>( \frac{n-1}{3} + 1 )</td>
<td>( \frac{n-1}{3} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Nature of ( n )</th>
<th>( e_f(0) )</th>
<th>( e_f(1) )</th>
<th>( e_f(2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n \equiv 0, 3 \pmod{6} )</td>
<td>( \frac{n}{3} )</td>
<td>( \frac{n}{3} )</td>
<td>( \frac{n}{3} )</td>
</tr>
<tr>
<td>( n \equiv 1 \pmod{6} )</td>
<td>( \frac{n+2}{3} - 1 )</td>
<td>( \frac{n+2}{3} )</td>
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</tr>
<tr>
<td>( n \equiv 2 \pmod{6} )</td>
<td>( \frac{n+1}{3} )</td>
<td>( \frac{n+1}{3} - 1 )</td>
<td>( \frac{n+1}{3} )</td>
</tr>
<tr>
<td>( n \equiv 4 \pmod{6} )</td>
<td>( \frac{n-1}{3} )</td>
<td>( \frac{n-1}{3} )</td>
<td>( \frac{n-1}{3} + 1 )</td>
</tr>
<tr>
<td>( n \equiv 5 \pmod{6} )</td>
<td>( \frac{n-1}{3} - 1 )</td>
<td>( \frac{n-1}{3} )</td>
<td>( \frac{n-1}{3} )</td>
</tr>
</tbody>
</table>

**Theorem 8.** The graph \( (P_n; S_2) \) is quotient-3 cordial.

**Proof.** Let \( V(G) = \{u_i, v_i, w_{ij} : 1 \leq i \leq n, 1 \leq j \leq 2\} \)
\( E(G) = \{[u_iu_{i+1} : 1 \leq i \leq n - 1] \cup [u_iv_i : 1 \leq i \leq n] \cup [v_iw_{ij} : 1 \leq i \leq n, 1 \leq j \leq 2] \} \)
Here \( |V(G)| = 4n, |E(G)| = 4n - 1. \)
Define \( f : V(G) \to Z_4 - \{0\} \) by
\( f(u_i) = 2, 1 \leq i \leq n \)
\( f(v_i) = 1, 1 \leq i \leq n \)
Labeling of \( w_{ij}, 1 \leq i \leq n, 1 \leq j \leq 2 \) is given below.
**Case (i):** When \( n \equiv 0 \pmod{3} \)
Without loss of generality, first set of \( \left( \frac{n}{3} \right) \) vertices by the label 1,
next set of \( \left( \frac{2n}{3} \right) \) vertices by the label 2 and the remaining set of \( \left( \frac{4n}{3} \right) \) vertices by the label 3.
**Case (ii):** When \( n \equiv 1 \pmod{3} \)
Without loss of generality, first set of \( \left( \frac{n+2}{3} \right) \) vertices by the label 1, next set of \( \left( \frac{n-1}{3} \right) \) vertices by the label 2 and the remaining set of \( \left( \frac{4n-1}{3} \right) \) vertices by the label 3.

**Case (iii):** When \( n \equiv 2 \pmod{3} \)

Without loss of generality, first set of \( \left( \frac{n+1}{3} \right) \) vertices by the label 1, next set of \( \left( \frac{n-2}{3} \right) \) vertices by the label 2 and the remaining set of \( \left( \frac{4n+1}{3} \right) \) vertices by the label 3.

The following table 3 gives that the graph \( (P_n; S_2) \) is quotient-3 cordial.

<table>
<thead>
<tr>
<th>Nature of ( n )</th>
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<td>( \frac{4n}{3} )</td>
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</tr>
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<td>( n \equiv 1 \pmod{3} )</td>
<td>( \frac{4n-1}{3} + 1 )</td>
<td>( \frac{4n-1}{3} )</td>
<td>( \frac{4n-1}{3} )</td>
</tr>
<tr>
<td>( n \equiv 2 \pmod{3} )</td>
<td>( \frac{4n+1}{3} )</td>
<td>( \frac{4n+1}{3} - 1 )</td>
<td>( \frac{4n+1}{3} - 1 )</td>
</tr>
</tbody>
</table>

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<th>Nature of ( n )</th>
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<tr>
<td>( n \equiv 0 \pmod{3} )</td>
<td>( \frac{4n}{3} )</td>
<td>( \frac{4n}{3} - 1 )</td>
<td>( \frac{4n}{3} )</td>
</tr>
<tr>
<td>( n \equiv 1 \pmod{3} )</td>
<td>( \frac{4n-1}{3} )</td>
<td>( \frac{4n-1}{3} )</td>
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</tr>
<tr>
<td>( n \equiv 2 \pmod{3} )</td>
<td>( \frac{4n+1}{3} )</td>
<td>( \frac{4n+1}{3} - 1 )</td>
<td>( \frac{4n+1}{3} - 1 )</td>
</tr>
</tbody>
</table>

**Theorem 9.** The graph \( (P_n; C_3) \) is quotient-3 cordial.

**Proof.** Let \( V(G) = \{u_i, v_i, w_{ij} : 1 \leq i \leq n, 1 \leq j \leq 2\} \)
\( E(G) = \{[(u_iu_{i+1}) : 1 \leq i \leq n-1] \cup [(u_iv_i) : 1 \leq i \leq n] \cup [(v_iw_{ij}) : 1 \leq i \leq n, 1 \leq j \leq 2] \cup [(w_{ij}w_{ij+1}) : 1 \leq i \leq n, j = 1]\} \)

Here \( |V(G)| = 4n, |E(G)| = 5n - 1 \)

Define \( f : V(G) \to \mathbb{Z}_4 - \{0\} \)

**Case (i):** When \( n \equiv 0 \pmod{3} \)

- \( f(u_i) = 2, 1 \leq i \leq n \)
- \( f(v_i) = 1, 1 \leq i \leq n \)
- \( f(w_{ij}) = 1, 1 \leq i \leq \frac{n}{3}, j = 1 \)
- \( f(w_{ij}) = 3, 1 \leq i \leq \frac{n}{3}, j = 2 \)
- \( f(w_{ij}) = 3, \frac{n}{3} + 1 \leq i \leq \frac{2n}{3}, 1 \leq j \leq 2 \)
- \( f(w_{ij}) = 3, \frac{2n}{3} + 1 \leq i \leq n, j = 1 \)
- \( f(w_{ij}) = 2, \frac{2n}{3} + 1 \leq i \leq n, j = 2 \)

\( \square \)
Case (ii): When \( n \equiv 1 \pmod{3} \)

\[
\begin{align*}
f(u_i) &= 2, \ 1 \leq i \leq n \\
f(v_i) &= 1, \ 1 \leq i \leq n \\
f(w_{ij}) &= 1, \ 1 \leq i \leq \frac{n-1}{3}, \ j = 1 \\
f(w_{ij}) &= 3, \ 1 \leq i \leq \frac{n-1}{3}, \ j = 2 \\
f(w_{ij}) &= 3, \ \frac{n+2}{3} \leq i \leq \frac{2n+1}{3}, \ 1 \leq j \leq 2 \\
f(w_{ij}) &= 3, \ \frac{2n+4}{3} \leq i \leq n, \ j = 1 \\
f(w_{ij}) &= 2, \ \frac{2n+4}{3} \leq i \leq n, \ j = 2
\end{align*}
\]

Case (iii): When \( n \equiv 2 \pmod{3} \)

\[
\begin{align*}
f(u_i) &= 2, \ 1 \leq i \leq n \\
f(v_i) &= 1, \ 1 \leq i \leq n-1 \\
f(v_i) &= 2, \ i = n \\
f(w_{ij}) &= 3, \ 1 \leq i \leq \frac{n+1}{3}, \ j = 1 \\
f(w_{ij}) &= 3, \ 2 \leq i \leq \frac{n+1}{3}, \ j = 2 \\
f(w_{ij}) &= 3, \ \frac{n+4}{3} \leq i \leq \frac{2n-1}{3}, \ 1 \leq j \leq 2 \\
f(w_{ij}) &= 3, \ \frac{2n+2}{3} \leq i \leq n-1, \ j = 1 \\
f(w_{ij}) &= 2, \ \frac{2n+2}{3} \leq i \leq n-1, \ j = 2
\end{align*}
\]

The following table 4 gives the quotient-3 cordial for \((P_n; C_3)\):

<table>
<thead>
<tr>
<th>Nature of ( n )</th>
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<tbody>
<tr>
<td>( n \equiv 0 \pmod{3} )</td>
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<td>( \frac{4n}{3} )</td>
</tr>
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<td>( \frac{4n-1}{3} )</td>
<td>( \frac{4n-1}{3} )</td>
<td>( \frac{4n-1}{3} + 1 )</td>
</tr>
<tr>
<td>( n \equiv 2 \pmod{3} )</td>
<td>( \frac{4n+1}{3} )</td>
<td>( \frac{4n+1}{3} )</td>
<td>( \frac{4n+1}{3} - 1 )</td>
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</thead>
<tbody>
<tr>
<td>( n \equiv 0 \pmod{3} )</td>
<td>( \frac{3n}{3} )</td>
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<td>( \frac{3n-1}{3} )</td>
<td>( \frac{3n-1}{3} )</td>
</tr>
</tbody>
</table>
Illustration 10. A quotient-3 cordial labeling of \((P_7; C_3)\) is given in figure 2

\[ \begin{array}{cccccccc}
1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array} \]

Figure 2:

4 Conclusion

It is very interesting to find out graphs or graph families which are quotient-3 cordial. In this paper, a quotient-3 cordial for some path related graphs has been found. The quotient-3 cordial labeling of some more graphs and graph families shall be explored further in future.

5 Acknowledgements

Register our heartful thanks to the referees who offered valuable suggestions and feedback.

References


