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ON $\alpha\omega$-CLOSED SETS IN TOPOLOGICAL SPACES

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Abstract: In this paper, we introduce the notion of $\alpha\omega$-closed sets in topological spaces and investigate some of their properties. Further, we introduce and study the concept of $\alpha\omega$-continuous functions.

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1. Introduction

Njastad [7] introduced the concept of α-closed sets in topological spaces. The notion of ω-closed sets are introduced by Sundaram and Sheik John [9] and recently Benchalli et.al [1] studied $\omega\alpha$-closed sets in topological spaces. Levine

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[5] introduced the concept of generalized closed sets in topological spaces. Gnanambal [4] introduced generalized pre closed sets in topological spaces. The notation of $g^\sharp s$-closed sets in topological spaces were inroduced by Veera Kumar[11].

In this paper, we introduce a new class of notion namely, $\alpha\omega$-closed sets for topological spaces. Further we introduce and study $\alpha\omega$-continuous map.

2. Preliminaries

Throughout this paper (X, τ) and (Y, σ) represent topological spaces. For a subset A of a space (X, τ), $cl(A)$ and $int(A)$ denote the closure of A and the interior of A respectively. $P(X)$ denotes the power set of X.

We recall the following definitions, which are useful in the sequel.

Definition 2.1. A subset A of a space (X, τ) is called

1. a generalized closed (briefly g-closed) set [5] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ); the complement of a g-closed set is called a g-open set,
2. a generalized preregular-closed (briefly gpr-closed) set [4] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in (X, τ),
3. a $\omega(= \widehat{g})$-closed set [9,12] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is *semi*-open in (X, τ),
4. a $g^\sharp$-closed set [11] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is αg-open in (X, τ),

Definition 2.2. A function $f : (X, \tau) \to (Y, \sigma)$ is called

1. g-continuous [2] if $f^{-1}(V)$ is g-closed in (X, τ) for every closed set V of (Y, σ),
2. gpr-continuous [4] if $f^{-1}(V)$ is gpr-closed in (X, τ) for every closed set V of (Y, σ),
3. g^*-continuous [10] if $f^{-1}(V)$ is g^*-closed in (X, τ) for every closed set V of (Y, σ),
4. $\widehat{g}$-continuous [13] if $f^{-1}(V)$ is $\widehat{g}$-closed in (X, τ) for every closed set V of (Y, σ),

5. $g^\sharp$-continuous [11] if $f^{-1}(V)$ is $g^\sharp$-closed in (X, τ) for every closed set V of (Y, σ) and

3. $\alpha\omega$-Closed Sets

We introduce the following definition.

Definition 3.1. A subset A of (X, τ) is called a $\alpha\omega$-closed set if $\omega cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in (X, τ). The complement of an $\alpha\omega$-closed set is an $\alpha\omega$-open set

Theorem 3.2. *Every closed set is a $\alpha\omega$-closed set.*

Proof. Let A be an closed set in (X, τ), then $A = cl(A)$. Let $A \subseteq U$, U is α-open in (X, τ). Since A is closed, $\omega cl(A) \subseteq cl(A) \subseteq U$. This shows that A is $\alpha\omega$-closed set.

The converses in the above theorem are not true as can be seen by the following example.

Example 3.3. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{b\}, \{b, c\}\}$. Here $C(X, \tau) = \{X, \phi, \{a\}, \{a, c\}\}$ and $\alpha\omega C(X, \tau) = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$ and let $A = \{a, b\}$. Then A is not a closed. However A is a $\alpha\omega$-closed set.

Theorem 3.4. *Every ω-closed set is a $\alpha\omega$-closed.*

Proof. Let A be an ω-closed set in (X, τ) and U is be any α-open set in X such that $A \subseteq U$. Since every α-open set is semi-open. Since A is closed, $\omega cl(A) \subseteq cl(A) \subseteq U$. This shows that A is $\alpha\omega$-closed set in $\{X, \tau\}$.

The converses in the above theorem are not true as can be seen by the following example.

Example 3.5. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{b\}, \{b, c\}\}$. Here $C(X, \tau) = \{X, \phi, \{a\}, \{a, c\}\}$ and $\alpha\omega C(X, \tau) = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$ and let $A = \{a, b\}$. Then A is a $\alpha\omega$-closed set but it is not ω-closed set.

Theorem 3.6. *Every $g^\sharp$-closed set is a $\alpha\omega$-closed.*

Proof. Let A be an $g^\sharp$-closed set in (X, τ) and U is be any α-open set in X such that $A \subseteq U$. Since every α-open set is $g\alpha$-open. Since A is closed, $\omega cl(A) \subseteq cl(A) \subseteq U$. This shows that A is $\alpha\omega$-closed set in $\{X, \tau\}$.

The converses in the above theorem are not true as can be seen by the following example.

Example 3.7. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a, b\}\}$. Here $g^\sharp C(X, \tau) = \{X, \phi, \{c\}, \{a, c\}, \{b, c\}\}$ and $\alpha\omega C(X, \tau) = P(X)$ and let $A = \{a\}$. Then A is not a $g^\sharp$-closed set. However A is a $\alpha\omega$-closed set.

Theorem 3.8. *Union of two $\alpha\omega$-closed sets are $\alpha\omega$-closed set.*

Proof. Let A and B be two $\alpha\omega$-closed sets, Let G be any α open set in $\{X, \tau\}$, such that $A \cup B \subseteq G$. Then $A \subseteq G$ and $B \subseteq G$. Since A and B are $\alpha\omega$-closed set, $\omega cl(A) \subseteq G$ and $\omega cl(B) \subseteq G$.Therefore $\omega cl(A) \cup \omega cl(B) = \omega cl(A \cup B) \subseteq G$. Hence $A \cup B$ is $\alpha\omega$-closed set.

Theorem 3.9. *If a subset A and X is $\alpha\omega$-closed set in $\{X, \tau\}$ then $\alpha\omega cl(A) - A$ does not contain any empty α closed sets in $\{X, \tau\}$.*

Proof. Suppose A is $\alpha\omega$-closed set and F be a non empty α closed subset of $\omega cl(A) - A$. Then $F \subseteq \omega cl(A) \cap (X - A)$. Since $(X - A)$ is α open and A is $\alpha\omega$-closed set. $\omega cl(A) \subseteq (X - A)$ therefore $F \subseteq (X - \omega cl(A))$. Thus $F \subseteq \omega cl(A) \cap (X - \omega cl(A)) = \phi$. This implies that $F = \phi$. Thus $\omega cl(A) - A$ does not contain any non empty $\alpha\omega$-closed.

Theorem 3.10. *If A is $\alpha\omega$-closed set in X and $A \subseteq B \subseteq \alpha\omega cl(A)$ then B is also $\alpha\omega$-closed set in X.*

Proof. Suppose A is $\alpha\omega$-closed set in X. Let $B \subseteq U$ such that U is α open set in X. Since $A \subseteq B$, we have $A \subseteq U$. Since A is $\alpha\omega$-closed and $\omega cl(B) \subseteq \omega cl(\omega cl(A)) = \omega cl(A) \subseteq U$. Therefore $\omega cl(B) \subseteq U$. Hence B is $\alpha\omega$-closed set in X.

The converse of the above theorem need not be true by the following example.

Example 3.11. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a, b\}\}$. Here $\alpha\omega C(X, \tau) = P(X)$ and the set $A = \{b\}$ and $B = \{b, c\}$. Such that A and B are $\alpha\omega$-closed sets but $A \subseteq B \subset \alpha\omega cl(A)$.

Theorem 3.11. *Let A be $\alpha\omega$-closed set in X. Then A is ω-closed iff $\omega cl(A) - A = \phi$ is α closed.*

Proof. Suppose A is ω closed. Then $\omega cl(A) = A$ and so $\omega cl(A) - A = \phi$, which is α closed. Conversely $\omega cl(A) - A$ is α closed. Then $\omega cl(A) - A = \phi$, Since A is $\alpha\omega$-closed set in X. That is $\omega cl(A) - A$ or A is ω closed.

Theorem 3.12. *Let A is α-open and $\alpha\omega$-closed set then A is ω closed.*

Proof. Since $A \subseteq A$ and A is α open and $\alpha\omega$ closed, we have $\omega cl(A) \subseteq A$. Thus $\omega cl(A) = A$. Hence A is ω closed set in X.

Theorem 3.13. *A set A is $\alpha\omega$-open in (X, τ) iff $F \subseteq \omega int(A)$ whenever F is α closed in (X, τ) and $F \subseteq A$.*

Proof. Suppose $F \subseteq \omega int(A)$ where F is α- closed and $F \subseteq A$. Let $X-A \subseteq G$ where G is α open in (X, τ). Then $G \subseteq X-G$ and $X-G \subseteq \omega int(A)$. Thus $X - A$ is $\alpha\omega$-closed set in (X, τ). Hence A is $\alpha\omega$-open in (X, τ).

Conversely, Suppose that A is $\alpha\omega$-open in (X, τ). $F \subseteq A$ and F is α-closed in (X, τ). Then $X - F$ is α-open and $X - A \subseteq X - F$. Therefore $\omega cl(X - A) \subseteq X - F$. But $\omega cl(X - A) = X - \omega int(A)$. Hence $F \subseteq \omega int(A)$.

Theorem 3.14. *A subset A is $\alpha\omega$-open in (X, τ) iff $G = X$ whenever G is α open and $\omega int(A) \cup (X - G) \subseteq G$.*

Proof. Let A be $\alpha\omega$-open. G be α-open and $\omega int(A) \cup (X - A) \subseteq G$. This gives $X - G \subseteq (X - \omega int(A)) \cap (X - (X - A)) = X - \omega int(A) - (X - A) = \omega cl(X - A) - (X - A)$. Since $X - A$ is $\alpha\omega$-closed and $X - G$ is α-closed. Then by Theorem 3.13 it follows that $X - G = \phi$. Therefore $X = G$.

Conversely, Suppose F is α-closed and $F \subseteq A$. Then $\omega int(A) \cup (X - A) \subseteq (A) \cup (X - F)$. It follows that $\omega int(A) \cup (X - F) = X$ and hence $F \subseteq \omega int(A)$. Therefore A is $\alpha\omega$-open in (X, τ).

4. $\alpha\omega$-Continuity

We introduce the following definition.

Definition 4.1. A function $f : (X, \tau) \to (Y, \sigma)$ is called a $\alpha\omega$-continuous if $f^{-1}(v)$ is a $\alpha\omega$-closed set of (X, τ) for every closed set v of (Y, σ).

Theorem 4.2. *Every continuous map is $\alpha\omega$-continuous.*

Proof. Let v be a closed set of (Y, σ). Since f is continuous and $f^{-1}(v)$ is closed in (X, τ). But every closed set is $\alpha\omega$-closed set. Hence $f^{-1}(v)$ is $\alpha\omega$-closed set in (X, τ). Thus f is $\alpha\omega$-continuous.

The converse need not be true by the following example.

Example 4.3. Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{b\}, \{b, c\}\}$ and $Y = \{a, b, c\}$, $\sigma = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = a, f(b) = b, f(c) = c$. Then $V = \{a, b\}$ in (Y, σ). $F^{-1}\{a, b\} = \{a, b\}$. Therefore v is $\alpha\omega$-continuous but not continuous.

Theorem 4.4. *Every ω-continuous map is $\alpha\omega$-continuous.*

Proof. Let v be a closed set of (Y, σ). Since $f^{-1}(v)$ is ω closed in (X, τ). But every ω closed set is $\alpha\omega$-closed set. Hence $f^{-1}(v)$ is $\alpha\omega$-closed set in (X, τ). Thus f is $\alpha\omega$-continuous.

The converse need not be true by the following example.

Example 4.5. Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{b\}, \{b, c\}\}$ and $Y = \{a, b, c\}$, $\sigma = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = a, f(b) = b, f(c) = c$. Then $V = \{a, b\}$ in (Y, σ). $F^{-1}\{a, b\} = \{a, b\}$. Therefore v is $\alpha\omega$-continuous but not ω-continuous.

Theorem 4.6. *Every $g^\sharp$-continuous map is $\alpha\omega$-continuous.*

Proof. Let v be a closed set of (Y, σ). Since $f^{-1}(v)$ is $g^\sharp$ closed in (X, τ). But every $g^\sharp$ closed set is $\alpha\omega$-closed set. Hence $f^{-1}(v)$ is $\alpha\omega$-closed set in (X, τ). Thus f is $\alpha\omega$-continuous.

The converse need not be true by the following example.

Example 4.7. Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{b\}, \{b, c\}\}$ and $Y = \{a, b, c\}$, $\sigma = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = a, f(b) = b, f(c) = c$. Then $V = \{a, b\}$ in (Y, σ). $F^{-1}\{a, b\} = \{a, b\}$. Therefore v is $\alpha\omega$-continuous but not $g^\sharp$-continuous.

Theorem 4.8. *A mapping $f : (X, \tau) \to (Y, \sigma)$ is called $\alpha\omega$-continuous if and only if $f^{-1}(v)$ is a $\alpha\omega$ closed set in (X, τ) for every closed set U in (Y, σ).*

Proof. Let U be any closed set of (Y, σ). By assumption $f^{-1}(U^c) = X - f^{-1}(U)$ is $\alpha\omega$ open in X, so $f^{-1}(U)$ is $\alpha\omega$-closed set in X.

The converse can be true in the same way.

Theorem 4.9. *A mapping $f : (X, \tau) \to (Y, \sigma)$ is called $\alpha\omega$-closed if and only if subset S of Y and for each open set U containing $f^{-1}(S)$ there is a $\alpha\omega$ open set U containing $f^{-1}(S)$ there is a $\alpha\omega$ open set V of Y such that $S \subseteq V$ and $f^{-1}(V) \subseteq U$.*

Proof. Suppose f is $\alpha\omega$ closed. Let S be a subset of Y and U be an open set of X such that $f^{-1}(S) \subseteq U$ then $V = Y - f(X - U)$ is an $\alpha\omega$ open set containing S such that $f^{-1}(V) \subseteq U$.

Conversely, suppose that F is a closed sets of X. Then $f^{-1}(Y - f(F) \subseteq X - F$ and $X - F$ is open. By hypothesis there is an $\alpha\omega$ open set V of Y such that $Y - f(F) \subseteq V$ and $f^{-1}(V) \subseteq X - F$. Therefore, $F \subseteq f^{-1}(V)$. Hence $Y - V \subseteq f(F) \subseteq f(X - f^{-1}(V)) \subseteq Y - V$ which implies $f(F) = Y - V$. Since $Y - V$ is $\alpha\omega$ closed, f(F) is $\alpha\omega$ closed and thus f is $\alpha\omega$ closed map.

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