PERRIN GRACEFUL GRAPHS

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Abstract

In this paper we prove that the star graph \(K_{1,n}\), the bistar graph \(B_{n,n}\), \(P_n^+\), \(C_n^+\), \(< K_{1,n}; 4 >\) are Perrin graceful graphs.

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1 Introduction

In this paper, a graph means it represents simple, finite, connected and undirected graph \(G = (V(G), E(G))\) with \(p\) vertices and \(q\) edges. A path of length \(n - 1\) is denoted by \(P_n\). A cycle of length \(n\) is denoted by \(C_n\). We denote \(G^+\) is a graph obtained from the graph \(G\) by attaching a pendant vertex to each vertex of \(G\). We recall some basic definitions and results which serve as prerequisites for this paper.
2 Basic Definitions

Definition 1. The graph $K_{1,n}^m$ is obtained from the path $P_m$ (path of length $m-1$) with $n$-pendant edges are attached at each end vertices of $P_m$.

For terms not defined here, standard terminology and notations related to graph theory we refer to Harary [4] while for number theory we refer to Burton [1]. A survey on graceful labelings can be found in Gallian [2]. Rosa [5] introduced $\beta$-labeling (or $\beta$-valuation) and later Golomb [3] named it as graceful labeling.

Definition 2. A graph $G = (V(G), E(G))$ is said to be graceful if $f : V(G) \rightarrow \{0, 1, 2, ..., q\}$ is an injection and the induced function $f^* : E(G) \rightarrow \{1, 2, ..., q\}$ defined by $f^*(e = uv) = |f(u) - f(v)|$ is a bijection. Here $f$ is called a graceful labeling of $G$.

Definition 3. The Perrin sequence of numbers $\{P_n\}$ can be defined by the linear recurrence relation satisfying the following conditions:

$P_1 = 3, P_2 = 0, P_3 = 2, \text{and } P_n = P_{n-2} + P_{n-3}$, if $n > 3$

Thus, $P_n$ generates the infinite sequence of integers in the following order:

$3, 0, 2, 3, 2, 5, 5, 7, 10, 12, ...$

We denote $P_i$ be the $i^{th}$ term of the Perrin sequence. We always assume that $P_0 = 0$.

In this paper, we combined the Perrin sequence and graceful labeling, we introduced a new graph called as Perrin graceful graph.

Definition 4. A function $f$ is called a Perrin graceful labeling of a graph $G = (V(G), E(G))$, if $f : V(G) \rightarrow \{P_0, P_1, P_2, ..., P_q\}$, is injective and the induced function $f^* : E(G) \rightarrow \{P_1, P_2, P_3, ..., P_q\}$ defined by $f^*(e = uv) = |f(u) - f(v)|$ is bijective. A graph which admits Perrin graceful labeling is called a Perrin graceful graph.
3 Main Results

Theorem 5. The star graph $K_{1,n}$ is Perrin graceful for every integer $n \geq 2$.

Proof. Let $G = K_{1,n}$ with $V(G) = \{u, u_1, u_2, ..., u_n\}$ be the vertex set of $G$, where $u$ be a central vertex and $u_i's$ be the pendant vertices adjacent to $u$.

Then, $|V(G)| = n + 1$ and $|E(G)| = n$.

Define $f : V(G) \rightarrow \{P_0, P_1, ..., P_n\}$ by

$f(u) = 0$, $f(u_i) = P_i$, $1 \leq i \leq n$.

We claim that the edge labels are distinct.

Let $E = f^*(E(G))$. Notice that

$E = \{f^*(uu_i) : 1 \leq i \leq n\}$, $E = \{|f(u) - f(u_i)| : 1 \leq i \leq n\}$,

$E = \{|f(u) - f(u_1)|, |f(u) - f(u_2)|, ..., |f(u) - f(u_n)|\}$,

$E = \{P_1, P_2, ..., P_n\}$.

Thus, the edge labels are distinct.

Therefore, $K_{1,n}$ admits Perrin graceful labeling.

\[ \square \]

Example 6. The star graph $K_{1,3}$ is a Perrin graceful graph is shown in Fig.1

![Fig.1](image)

Theorem 7. The Bistar graph $B_{n,n}$ is Perrin graceful graph.

Proof. Let $G = B_{n,n}$ with $V(G) = \{u, v, u_i, v_j : 1 \leq i, j \leq n\}$ be the vertex set of $G$, where $u$ and $v$ are the central vertices and $u_i$, $v_j$ are the pendant vertices adjacent to $u$ and $v$ respectively.

Then $|V(G)| = 2n + 2$ and $|E(G)| = 2n + 1$.

Define $f : V(G) \rightarrow \{P_0, P_1, ..., P_{2n+1}\}$ by

$f(u) = P_0$, $f(v) = P_2$, $f(u_i) = P_1$,

$f(u_i) = P_{i+1}$, $2 \leq i \leq n$, $f(v_j) = P_{n+1+j}$, $1 \leq j \leq n$.

We claim that the edge labels are distinct.

$E_1 = \{f^*(uv)\}$, $E_1 = \{|f(u) - f(v)|\}$, $E_1 = \{P_2\}$.
$E_2 = \{f^*(uu_1)\}, E_2 = \{|f(u) - f(u_1)|\}, E_2 = \{P_1\}.$
$E_3 = \{f^*(uu_i) : 2 \leq i \leq n\}, E_3 = \{|f(u) - f(u_i)| : 2 \leq i \leq n\},$
$E_3 = \{|f(u) - f(u_2)|, |f(u) - f(u_3)|, ..., |f(u) - f(u_n)|\},$
$E_3 = \{P_3, P_4, ..., P_{n+1}\}.$
$E_4 = \{f^*(v_j) : 1 \leq j \leq n\}, E_4 = \{|f(v) - f(v_j)| : 1 \leq j \leq n\},$
$E_4 = \{|f(v) - f(v_1)|, |f(v) - f(v_2)|, ..., |f(v) - f(v_n)|\},$
$E_4 = \{P_{n+2}, P_{n+3}, ..., P_{2n+1}\}.$
Let $E = f^*(E(G))$, then $E = E_1 \cup E_2 \cup E_3 \cup E_4,$
$E = \{P_1, P_2, ..., P_{2n+1}\}.$
Thus, the edge labels are distinct.
Therefore, $B_{n,n}$ admits Perrin graceful labeling. \hfill \Box

**Example 8.** The Perrin graceful labeling of $B_{2,2}$ is shown in Fig.2

![Fig.2](image)

**Theorem 9.** $P_n^+$ is a Perrin graceful graph for every integer $n \geq 2.$

**Proof.** Let $G = P_n^+$ with $V(G) = \{u_1, u_2, ..., u_{n+1}\} \cup \{v_1, v_2, ..., v_{n+1}\}$ be the vertex set of $G.$
Then $|V(G)| = 2n + 2$ and $|E(G)| = 2n + 1.$
Define $f : V(G) \rightarrow \{P_0, P_1, ..., P_{2n+1}\}$ by
$f(u_i) = P_{2i}, 1 \leq i \leq n, f(u_{n+1}) = P_0,$
$f(v_1) = P_{2n+1}, f(v_{j+1}) = P_{2j-1}, 1 \leq j \leq n.$
We claim that the edge label are distinct. let
$E_1 = \{f^*(u_iu_{i+1}) : 1 \leq i \leq n - 1\},$
$E_1 = \{|f(u_i) - f(u_{i+1})| : 1 \leq i \leq n - 1\},$
$E_1 = \{|f(u_1) - f(u_2)|, |f(u_2) - f(u_3)|, ..., |f(u_{n+1}) - f(u_n)|\},$
$E_1 = \{P_1, P_3, P_5, ..., P_{2n-3}\}.$
$E_2 = \{f^*(uu_{n+1})\}, E_2 = \{|f(u) - f(u_{n+1})|\}, E_2 = \{P_{2n}\}.$
$E_3 = \{f^*(u_1v_1)\}, E_3 = \{|f(u_1) - f(v_1)|\}, E_3 = \{P_{2n+1}\}.$
$E_4 = \{f^*(u_iv_j) : 2 \leq i, j \leq n\}, E_4 = \{|f(u_i) - f(v_j)| : 2 \leq i, j \leq n\},$
$E_4 = \{|f(u_2) - f(v_2)|, |f(u_3) - f(v_3)|, \ldots, |f(u_n) - f(v_n)|\},$

$E_4 = \{P_2, P_4, P_6, \ldots, P_{2n-2}\}.$

$E_5 = \{f^*(u_{n+1}v_{n+1})\}, E_5 = \{|f(u_{n+1}) - f(u_{n+1})|\}, E_5 = \{P_{2n-1}\}.$

Let $E = f^*(E(G))$, then $E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$.

Thus, the edge labels are distinct. Therefore, $P_n^+$ admits Perrin graceful labeling. \hfill \Box

**Example 10.** The Perrin graceful labeling of $P_4^+$ is shown in Fig.3.

![Fig3](image)

**Theorem 11.** $G_n^+$ is a Perrin graceful graph for every integer $n \geq 3$.

**Proof.** Let $G = G_n^+$ with $V(G) = \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$ be the vertex set of $G$.

Then $|V(G)| = 2n$ and $|E(G)| = 2n$.

Define $f : V(G) \rightarrow \{P_0, P_1, \ldots, P_{2n}\}$ by

$f(u_i) = P_{2i}, 1 \leq i \leq n.$

$f(v_j+1) = P_{2j-1}, 1 \leq j \leq n - 1.$

$f(v_1) = P_{2n-1}.$

We claim that the edge label are distinct. Let

$E_1 = \{f^*(u_iu_{i+1}) : 1 \leq i \leq n - 1\},$

$E_1 = \{|f(u_i) - f(u_{i+1})| : 1 \leq i \leq n - 1\},$

$E_1 = \{|f(u_1) - f(u_2)|, |f(u_2) - f(u_3)|, \ldots, |f(u_{n-1}) - f(u_n)|\},$

$E_1 = \{P_1, P_3, P_5, \ldots, P_{2n-3}\}.$

$E_2 = \{f^*(u_nu_1)\}, E_2 = \{|f(u_n) - f(u_1)|\}, E_2 = \{P_{2n}\}.$

$E_3 = \{f^*(u_iv_j) : 2 \leq i, j \leq n\}, E_3 = \{|f(u_i) - f(v_j)| : 2 \leq i, j \leq n\},$

$E_3 = \{|f(u_2) - f(v_2)|, |f(u_3) - f(v_3)|, \ldots, |f(u_n) - f(v_n)|\},$

$E_3 = \{P_2, P_4, P_6, \ldots, P_{2n-2}\}.$

$E_4 = \{f^*(u_1v_1)\}, E_4 = \{|f(u_1) - f(v_1)|\}, E_4 = \{P_{2n-1}\}.$

Let $E = f^*(E(G))$, then $E = E_1 \cup E_2 \cup E_3 \cup E_4$. 

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$E = \{P_1, P_2, \ldots, P_{2n}\}$. Thus, the edge labels are distinct. Therefore, $C_n^+$ admits Perrin graceful labeling.

Example 12. The Perrin graceful labeling of $C_3^+$ is shown in Fig.4.

![Fig.4](image)

Theorem 13. $<K_{1,n}; m>$ is a Perrin graceful graph only for $m = 4$.

Proof. Let $G$ be a $<K_{1,n}; 4>$ graph with $V(G) = \{u_1, u_2, \ldots, u_n\} \cup \{u, v, x, y\} \cup \{v_1, v_2, \ldots, v_n\}$ be the vertex set of $G$. Then $|V(G)| = 2n + 4$ and $|E(G)| = 2n + 3$.

Define $f : V(G) \to \{P_0, P_1, \ldots, P_{2n+3}\}$ by

- $f(u) = P_0, f(v) = P_2, f(x) = P_1, f(y) = P_4, f(u_1) = P_3$,
- $f(u_i) = P_{i+3}, 2 \leq i \leq n, f(v_j) = P_{n+3+j}, 1 \leq j \leq n$.

We claim that the edge labels are distinct.

- $E_1 = \{f^*(uu_i) : 2 \leq i \leq n\}$,
- $E_2 = \{|f(u) - f(u_1)|, |f(u) - f(u_2)|, \ldots, |f(u) - f(u_n)|\}$,
- $E_3 = \{P_0, P_1, \ldots, P_{n+3}\}$,
- $E_4 = \{|f(x) - f(x)|, |f(x) - f(y)|, |f(y) - f(v)|, |f(u) - f(u_1)|\}$,
- $E_5 = \{P_2, P_3, P_4, P_5, P_6\}$,
- $E_6 = \{|f(v) - f(v_j)| : 1 \leq j \leq n\}$,
- $E_7 = \{|f(v) - f(v_1)|, |f(v) - f(v_2)|, \ldots, |f(v) - f(v_{2n+3})|\}$,
- $E_8 = \{P_{n+4}, P_{n+5}, \ldots, P_{2n+3}\}$.

Let $E = f^*(E(G))$, then $E = E_1 \cup E_2 \cup E_3$. $E = \{P_1, P_2, \ldots, P_{2n+3}\}$.

Thus, the edge labels are distinct. Therefore, $<K_{1,n}; 4>$ admits Perrin graceful labeling.

Example 14. The Perrin graceful labeling of $<K_{1,3}; 4>$ is shown in Fig.5.
4 Conclusion

In this paper we applied the concept of Perrin sequence of numbers and studied its behaviour on graceful labeling and proved that the star graph $K_{1,n}$, the bistar graph $B_{n,n}$, $P_n^+$, $C_n^+$, $< K_{1,n}; 4 >$ are Perrin graceful graphs. It is an interesting open question to identify the other graphs which admit Perrin graceful labeling.

References


