COMPLEX SOLUTIONS FOR TSUNAMI-ASCENDING INTO A RIVER AS A BORE

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Abstract

Tsunami is regarded as a natural hazard which causes severe damage to the coastal areas. In the vast tsunami research it was not much discussed about the propagation of tsunami waves upstream into rivers; Tsunamis are termed as destructive long waves since their wavelength is much larger when compared to the ocean depth across which it travels. The dispersive effects are insignificant while travelling in the open ocean and become significant when the propagating distance is very long. The potential of tsunami propagation in a coastal river and its destructiveness cannot be ignored. In this study, the KdV-Burger equation which was introduced by Johnson in 1972 along with the frictional term which accounts for the energy dissipation is solved using Modified Extended Direct Algebraic (MEDA) method.
and implemented in a computer algebraic system.

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**Key Words:** Tsunami Bore; KdV- Burgers equation; Energy dissipation; Modified Extended Direct Algebraic method

## 1 Introduction

The generation of giant tsunami waves which is considered as a first phase in the lifecycle of tsunami can be attributed to various geophysical processes such as earthquake, landslide, volcanic eruptions etc., while the second phase of propagation they are non-dispersive long waves. The last phase of run-up is highly complex which is caused due to the nonlinearity, dispersion and energy dissipation of the incoming tsunami waves.

Most mathematical models on tsunami propagation are based upon shallow water wave theory which poorly predicts the breaking of waves offshore. Ascending of tsunami into a river like an undular type bore or non-undular type bore, which were recognized as solutions for KdV Burgers equation was studied [1] The non-linear dispersive model based on the forced Korteweg de Vries model (fKdV) was developed from the Boussinesq shallow water model, which provided the possibility of observing tsunami generation by atmospheric disturbances; the fKdV is solved using explicit finite difference method [2].

![Figure 1: A snapshot of tsunami entering the Natori River, south of Sendai [13]](image)
Guycaputo [3] modeled the tsunami propagation in a channel of varying depth and width, using inhomogenous K-dV with different types of dissipation which are used in geophysical flows Rayleigh, Chezy and Reynolds damping.

2 Model

The surface profile above a fully developed Poiseuille channel flow was studied; the profile close to the kinematic wave was examined and was shown to satisfy an equation of the form,

\[ \eta_t + \eta \eta_x + \alpha \eta_{xx} = \beta \eta_{xx} \] (2.1)

where \( \eta(X, T) \) is the surface profile, the above equation is called Korteweg-de Vries -Burgers equation, as \( \alpha \to 0 \), the equation reduces to the Burgers equation and as \( \beta \to 0 \), it represents the K-dV equation [4].

Pelinovskii [5] proposed the control equation for tsunami waves as,

\[ \eta_t + \kappa(1 + \frac{3\eta}{2D}) + \eta_x(c_0 \frac{D^2}{6})\eta_{xxx} = -F(\eta) \] (2.2)

where \( \eta \) is the displacement, \( \kappa \) is long wave velocity, \( \sqrt{gD} \), D is depth and \( F(\eta) \) is a kind friction function.

Without neglecting the effect of the dissipation of energy, we have the KdV- Burgers equation as,

\[ \eta_t + \kappa \eta_x + \frac{3c_0^2}{2D} \eta \eta_x + \frac{c_0 D^2}{6} \eta_{xxx} = G \eta_{xx} \] (2.3)

where \( G \) is the coefficient of energy dissipation is of the form, \( \nu \) coefficient of dynamic viscosity, \( L \) is the wavelength and \( Q \) is constant, which was given by Chester [6] as

\[ Q = \frac{(1 - \frac{2}{3} F^2)}{3}; \quad G = \frac{Q \nu L^2}{D^2} \]

where \( F \) is a kind of Froude number given by \( F = U/\sqrt{gD} \), and \( U \) is the velocity of the bore.
Following [3] the following transformation of variables are made,

\[ \zeta = \left( \frac{3\kappa}{2D} \right) \eta; \quad s = \omega(x - \kappa t); \quad \tau = \omega^{-1/2}t; \quad \lambda = \omega^{-1/2}G; \quad \omega = \frac{c_0 D^2}{6} \]

which yields,

\[ \zeta_\tau + \omega^{3/2}\zeta_\zeta s + \omega^{9/2}\zeta sss = \omega^{5/2}G\zeta_{ss} \quad (2.4) \]

### 3 Analytic Method

The spread of tsunami wave showed in homogeneous and non-homogeneous ocean using the cellular automata[8] and transformation of wave fronts [7]. A modified F-expansion method was proposed by taking full advantage of F-expansion method and Riccati equations in seeking exact solutions of nonlinear partial equations [9]. A direct algebraic method to construct exact complex solutions of nonlinear partial differential equations by using some exact solutions of an auxiliary ordinary differential equation was found [10]. A modified version of the above said method was found and new complex solutions for nonlinear partial differential equations such as Burgers, KdV- Burgers, Coupled Burgers and two-dimensional Burgers equation [10].

Consider a partial differential equation in two independent variables given by,

\[ L(u, u_x, u_t, u_{xx}, ...) = 0 \quad (3.1) \]

Considering a travelling wave solution

\[ u(x, t) = u(z); \quad z = i(x + ct) \quad \text{or} \quad z = i(x - ct), i = \sqrt{-1} \]

then (3.1) becomes an ordinary differential equation,

\[ F(u, iu, -icu', -u'', ... ) = 0 \quad (3.3) \]

where \( u' = \frac{du}{dz} \)

Introducing the following ansatz,

\[ u(z) = c_0 + \sum_{j=1}^{N} (\lambda^j c_j + \lambda^{-j} d_j) \quad (3.4) \]

\[ \lambda' = b + \lambda^2 \quad (3.5) \]

where b is unknown to be determined, \( \lambda = \lambda(z), \lambda' = \frac{d\lambda}{dz} \). The
parameter N is obtained by balancing the highest order derivative term with the non-linear terms [11]. Substituting (3.4) in (3.3) along with (3.5) will give rise on to a system of algebraic equations with respect to $c_j, d_j, b$ and $c$ (where $j = 1, 2, ..., N$). By solving the system of equations we obtain $c_j, d_j, b, c_0$ and $c$. The general solution for (3.5) is as follows:

(i) if $b < 0$, then $\lambda = -i\sqrt{btanh(i\sqrt{b}z)}$ or $\lambda = -i\sqrt{bcoth(i\sqrt{b}z)}$ depending on the initial conditions.
(ii) if $b > 0$, then $\lambda = \sqrt{btanh(\sqrt{b}z)}$ or $\lambda = \sqrt{bcoth(\sqrt{b}z)}$ depending on the initial conditions.
(iii) if $b = 0$, then $\lambda = \frac{-1}{z}$ (3.6)

Substituting these in (3.6) according to the value of $b$ we obtain travelling wave solutions for (3.3). Now, transform $u(x, t) = u(z), \ z = i(x + ct)$ Eq.(2.4) becomes,

\[
\begin{align*}
&icu' + \omega^2 i u'' - \omega^2 i u''' + \omega^2 Gu'' = 0 \\
&icu' + \omega^2 i (u^2)' - \omega^2 i u''' + \omega^2 Gu'' = 0 \\
\end{align*}
\]

Integrating once,

\[
\begin{align*}
&icu' + \omega^2 i u^2 - \omega^2 i u'' + \omega^2 Gu' = 0 \quad (3.7) \\
\end{align*}
\]

Balancing the non-linear term and highest order derivative in the above equation we get, $N = 2$ and

\[
\begin{align*}
u(z) = c_0 + \lambda^{-1}d_1 + \lambda^{-2}d_2 + \lambda c_1 + \lambda^2 c_2 \quad (3.8) \\
\end{align*}
\]

Substituting (3.8) along with (3.5) in the (3.7) and letting the initial conditions to be that of a bore which travelled up the Natori river of south Sendai during the 2011- Tohuko Japan tsunami. The American Society of Civil Engineers (ASCE) surveyed the river [12], the bore propagation velocity was calculated as,

\[
v_j = \sqrt{gd_s (\frac{1}{2}(\frac{h_s}{d_s})^2 + \frac{1}{2}(\frac{h_s}{d_s}))} = 6.14ms^{-1}
\]

where $v_j$ is the jump velocity ($ms^{-1}$), $g$ is the acceleration due to gravity ($ms^{-2}$), in the front $d_s$ is the still water depth in front of the jump (m) and $h_s = h_j + d_s$ is the height of the bore, where $h_j$ is the
height of the hydraulic jump (m). During the field survey the still water depth was 1.21m and a hydraulic jump height of 1.21m was noted from the video footage. The coefficient of dynamic viscosity is taken to be $8.94 \times 10^{-4}$. Considering the above bore velocity and still water depth as initial conditions and assuming the wavelength to be 40m and dynamic viscosity of water is for Eq (3.7), we get $G, \omega$ as -0.08806 and 0.840710.

After applying the initial conditions, we get the following system of algebraic equations by equating each coefficient of $\lambda^j$ to zero,

- $2.748282083i_d^2b^2 + 0.3854251260i_d^2 = 0$
- $0.7708502520i_c d^2 - 0.9160940278ib^2d_1 + 0.1141373213bd_2 = 0$
- $-icd_2 + 0.7708502520ic_0d_2 - 3.664376111i_d b + 0.0570686668d_1 b + 0.3854251260i_d^2 = 0$
- $-0.9160940278id_1 b - icd_1 + 0.7708502520ic_0d_1 + 0.1141373213d_2 = 0$
- $-icc_0 + 0.7708502520ic_1 d_1 + 0.7708502520ic_2 d_2 - 0.9160940278id_2 + 0.3854251260ic_0^2 - 0.9160940278ic_2 b^2 + 0.0570686668d_1 - 0.0570686668c_1 b = 0$
- $-0.1141373213c_2 b - icc_1 + 0.7708502520ic_0 c_1 + 0.7708502520ic_2 d_1 - 0.9160940278ic_1 b = 0$
- $-0.0570686668c_1 - icc_2 + 0.7708502520ic_0 c_2 - 3.664376111ib c_2 + 0.3854251260ic_1^2 = 0$
- $-0.9160940278ic_1 - 0.1141373213c_2 + 0.7708502520ic_1 c_2 = 0$
- $-2.748282083ic_2 + 0.3854251260ic_2^2 = 0$

Solving for $b, c, c_0, c_1, c_2, d_1$ and $d_2$ and substituting for $\lambda(z)$ as $-\sqrt{bcot(\sqrt{b}z)}$ where $z = i(x - ct)$, we get the following complex solutions,

Case 1:

$$u(z) = -0.001660304673 - 0.1776801463i\lambda(z) + 7.130521332\lambda(z)^2 + 0.000006895531376 \frac{1}{\lambda(z)} + 1.073870210 \times \frac{10^{-8}}{\lambda(z)^2}$$

Case 2:

$$u(z) = -0.001106869783 + 0.00002758126304 \frac{1}{\lambda(z)} + 1.718192336 \times \frac{10^{-7}}{\lambda(z)^2}$$

Case 3:

$$u(z) = -0.001106869783 - 0.1776801463i\lambda(z) + 7.130521332\lambda(z)^2$$

Case 4:

$$u(z) = 0.002767174453 - 0.1776801463i\lambda(z) + 7.130521332\lambda(z)^2 + \ldots$$
\[0.000006895531376 \frac{i}{\lambda(z)} + 1.073870210 \times 10^{-8} \]

Case 5:
\[u(z) = 0.003320609343 + 0.00002758126304 \frac{i}{\lambda(z)} + 1.718192336 \times 10^{-7} \]

Case 6:
\[u(z) = 0.003320609343 - 0.1776801463i\lambda(z) + 7.130521332\lambda(z)^2\]

4 Discussion

There are a plenty of numerical techniques to handle a given mathematical model; due to the advancement of various analytical techniques for solving nonlinear differential equations we are able to get closed form of solutions for various mathematical models for which earlier numerical techniques had been used. This work sheds some light on tsunami water waves propagating upstream in a river or channel.

Figure 2: The surface profiles a, b, c, d, e and f are graphical representation of cases 1, 2, 3, 4, 5 and 6

Instead of solving Eqn. 2.4 directly, [3] simplified the problem
by another transformation and solved the resulting ordinary second order differential equation by using a numerical technique the phase plane method. By varying the parameter of viscosity $P$, they have observed that if $P$ is small, the bore takes undular type weak bore and if $P$ is greater the shape is that of a strong bore.

5 Conclusion

In this study, we have presented a graphical representation of some complex analytic solutions implemented in Maple 13, a computer algebraic system for the control equation of the tsunami waves. In Figure 2., the surface profiles a, b, c, d, e and f are graphical representation of cases 1, 2, 3, 4, 5 and 6. The surface profile of the bore is seen in which the profiles a, c and d, f appear to be same if taken as a pair, where as the profiles b and e are distinct. So, out of the six surface profiles obtained using MEDA the profiles corresponding to cases 2 and 5 give better results. On examining the above results we see the value of the constant $c_0$ plays one of the vital roles in determining the surface profile. Though we have assumed an approximate wavelength to calculate the energy dissipation term, if provided with the accurate wavelength of the bore entering the river, we can further improve the solution and solve the problem using analytical methods.

References


