Abstract

To monitor an electric power system by placing as few phase measurement units (PMUs) in the system as possible is closely related to the famous domination problem in graph theory. A set $S$ is a dominating set of $G$, if every vertex in $V/S$ is adjacent to a vertex in $S$. The domination number of $\gamma(G)$ is the minimum cardinality of dominating set of $G$. A set $S$ is a power dominating set of a graph $G = (V,E)$, if every vertex and every edge in the system is monitored following the observation rules of power monitoring system. The power domination number of a graph $G$ is the minimum cardinality of a power dominating set of $G$. In this paper, we investigate the power domination of the middle graph of path $P_n$, cycle $C_n$ and star $K_{1,n}$.

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tion number, Power dominating set, Middle graph.
1 Introduction

A set \( S \subseteq V(G) \) is a dominating set of graph \( G \), if every vertex in \( V/S \) is adjacent to a vertex in \( S \). The problem of finding minimum dominating set is an important problem that has been elaborately studied. The domination number \( \gamma(G) \) is the minimum cardinality of dominating set of \( G \).

The power domination problem arose in the context of monitoring the electric power networks. A power network contains a set of nodes and a set of edges connecting the nodes. It also accommodates a set of generators which supplies power, and a set of loads to which the power is directed to. Using the measurement devices, we need to measure all the state variables of the network in order to monitor a power network. A Phase Measurement Unit (PMU) is a measurement device placed on a node that has the ability to measure the voltage of the node and current phase of the edges connected to the node and to give warnings of system-wide failures. The ultimate goal is to install the minimum number of PMUs in order to monitor the whole system.

Haynes et al. remodeled this problem as a graph domination problem in [6]. Since the domination rules are iterative, this type of domination is different from the standard domination type problem. The propagation rules are derived from the Ohm’s and Kirchoff’s laws for an electric circuit.

Let \( G = (V, E) \) be a graph representing an electric power system, where a vertex represents an electrical node and an edge represents a transmission line joining two electrical nodes.

For a vertex \( v \) of \( G \), let \( N(v) \) denote the open neighborhood of \( v \). Let \( N(S) = \bigcup_{v \in S} N(v) - S \), where \( S \) is a subset of \( V(G) \). The closed neighborhood of \( N[S] \) of a subset \( S \) is the set \( N[S] = N(S) \cup S \). The maximum degree of the vertices of \( G \) is denoted by \( \Delta(G) \). The sub graph of \( G \) induced by \( X \) is denoted by \( G[X] \). For any real number \( n \), \( \lceil n \rceil \) denotes the integer greater than or equal to \( n \) and \( \lfloor n \rfloor \) denotes the integer less than or equal to \( n \).

For a connected graph \( G \) and a subset \( S \subseteq V(G) \), let \( M(S) \) denote the set monitored by \( S \). It is defined recursively as follows:

1. Domination step
   \[ M(S) \leftarrow S \cup N(S) \]
2. Propagation step
As long as there exists \( v \in M(S) \) such that \( N(v) \cap (V(G) - M(S)) = \{w\} \) set \( M(S) \leftarrow M(S) \cup \{w\} \).

A set \( S \) is called a power dominating set of \( G \) if \( M(S) = V(G) \) and the power domination number of \( G \), \( \gamma_p(G) \), is the minimum cardinality of a power dominating set of \( G \). For any graph \( G \), \( 1 \leq \gamma_p(G) \leq \gamma(G) \).

While domination has been studied elaborately (see \([4, 7, 8]\))

power domination is introduced and studied very recently (see \([1, 2, 6]\)). In particular the power domination decision problem has been shown to be NP-complete, even when restricted to bipartite graphs and chordal graphs [6] or even split graphs [9]. The upper bounds for \( \gamma_p(G) \) where \( G \) is a cylinder, a torus, and a generalized Petersen graph were obtained by Barrera and Ferrero [3]. Later Dorfling and Henning presented the power domination number for grid graphs [5]. The upper bounds for the power dominating sets of block graphs and claw free graphs were studied in [10] and [11] respectively. In this paper, we investigate the power domination of middle graph of path \( P_n \), cycle \( C_n \) and \( K_{1,n} \).

**Observation 1.** [6] If \( G \) is a graph with maximum degree
at least 3, then \( G \) contains a \( \gamma_p(G) \)-set in which every vertex has
degree at least 3

**Definition 2.** The middle graph of a connected graph \( G \)
denoted by \( M(G) \) is the graph whose vertex set is \( V(G) \cup E(G) \)
where two vertices are adjacent if

(i) They are adjacent edges of \( G \) or

(ii) One is a vertex of \( G \) and the other is an edge incident with it.

In the next section we obtain the power domination number
\( \gamma_p(G) \) of the middle graph of path \( P_n \).

**2 Power Domination of the Middle Graph of \( P_n \)**

**Theorem 3.** For any positive integer \( n \), the power domination
number of \( M(P_n) = \left\lceil \frac{n-1}{3} \right\rceil \).
Proof. Let \( u_1, u_2, u_3, \ldots, u_n \) be the vertices of path \( P_n \) and let \( v_1, v_2, v_3, \ldots v_{n-1} \) be the added vertices corresponding to the edges \( e_1, e_2, e_3, \ldots e_{n-1} \) of \( P_n \) to obtain \( M(P_n) \). Then, 
\[ |V(M(P_n))| = 2n - 1 \] and 
\[ |E(M(P_n))| = 3n - 1. \]

Construct the set 
\[
S = \left\{ \begin{array}{ll}
\{ v_i : i \equiv 2 \text{ mod } 3 \} \cup \{ v_{n-1} \}, & \text{if } n = 3r - 1, r \text{ is an integer} \\
\{ v_i : i \equiv 2 \text{ mod } 3 \}, & \text{otherwise}
\end{array} \right\}
\]

where \( 1 \leq i \leq n \) with 
\[ |S| = \left\lceil \frac{n-1}{2} \right\rceil. \]

Since each vertex in \( V(M(P_n)) \) is either in \( N[S] \) or adjacent to a monitored vertex, \( S \) is a power dominating set for \( M(P_n) \). Thus 
\[ \gamma_p(M(P_n)) \leq \left\lceil \frac{n-1}{3} \right\rceil. \]

Let \( S \) be a \( \gamma_p \)-set for \( M(P_n) \).

As \( \Delta(M(P_n)) \geq 3 \), we observe that \( \deg(v) \geq 3 \) for all \( v \in S \). In particular every vertex in \( S \) lies on the path \( (v_1, v_2, v_3, \ldots, v_{n-1}) \) of \( M(P_n) \). Now suppose that \( \gamma_p(M(P_n)) < \left\lceil \frac{n-1}{3} \right\rceil. \)

Then there is a vertex \( v_i \in M(P_n) \) such that \( v_i \not\in S \) and 
\[ d(v_i, N(s)) = 1. \] So \( v_i \not\in N[S] \). This vertex will be observed only if at least one edge incident to it is observed. The edge \( u_i v_i \) is not observed unless if \( v_{i-1} u_i \) is observed. So \( v_{i-1} v_i \) is also not observed. The edge \( u_{i+1} v_{i+1} \) is not observed, since \( v_i u_{i+1} \) is not observed. So the edge \( v_i v_{i+1} \) is also not observed. Hence \( S \) is not a power dominating set. This completes the proof. \( \square \)

3 Power Domination of the Middle Graph of \( C_n \)

In this section we obtain the power domination number \( \gamma_p(G) \) of the middle graph of cycle \( C_n \).

**Theorem 4.** For any positive integer \( n \), the power domination number of \( M(C_n) = \left\lceil \frac{n}{3} \right\rceil. \)
Proof. Let \( u_1, u_2, u_3, \ldots, u_n \) be the vertices of cycle \( C_n \) and let \( v_1, v_2, v_3, \ldots, v_n \) be the added vertices corresponding to the edges \( e_1, e_2, e_3, \ldots, e_n \) of \( C_n \) to obtain \( M(C_n) \). Then, 
\[
|V(M(C_n))| = 2n \quad \text{and} \quad |E(M(C_n))| = 3n.
\]
Construct the set \( S = \{v_i : i \equiv 1 \mod 3\} \) where \( 1 \leq i \leq n \) with 
\[
|S| = \left\lceil \frac{n}{3} \rightceil.
\]
Since each vertex in \( V(M(C_n)) \) is either in \( N[S] \) or adjacent to a monitored vertex, \( S \) is a power dominating set for \( M(C_n) \). Thus 
\[
\gamma_p(M(C_n)) \leq \left\lceil \frac{n}{3} \right\rceil.
\]
Let \( S \) be a \( \gamma_p \)-set for \( M(C_n) \).

As \( \Delta(M(C_n)) = 4 \), we observe that \( \deg(v) = 4 \) for all \( v \in S \). In particular every vertex in \( S \) lies on the cycle \( (v_1, v_2, v_3, \ldots, v_n) \) of \( M(C_n) \). Now suppose that 
\[
\gamma_p(M(C_n)) < \left\lceil \frac{n}{3} \right\rceil.
\]
Then there is a vertex \( v_i \in M(C_n) \) such that \( v_i \not\in S \) and 
\[
d(v_i, N(s)) = 1. \quad \text{So} \quad v_i \not\in N[S]. \quad \text{This vertex will be observed only if at least one edge incident to it is observed. the edge} \ u_i,v_i \text{ is not observed unless if} \ v_i-1u_i \text{ is observed. So} \ v_i-1v_i \text{ is also not observed. The edge} \ u_{i+1}v_{i+1} \text{ is not observed, since} \ v_iu_{i+1} \text{ is not observed. So the edge} \ v_iv_{i+1} \text{ is also not observed. Hence} \ S \text{ is not a power dominating set. This completes the proof.} \]

4 Power Domination of the Middle Graph of \( K_{1,n} \)

In this section we obtain the power domination number \( \gamma_p(G) \) of the middle graph of \( K_{1,n} \).
Theorem 5. For any positive integer $n$, $\gamma_p(M(K_{1,n})) = 1$.

Proof. Let $u, u_1, u_2, u_3, \ldots, u_n$ be the vertices of star graph and let $v_1, v_2, v_3, \ldots, v_n$ be the added vertices corresponding to the edges $e_1, e_2, e_3, \ldots, e_n$ of $K_{1,n}$ to obtain $M(K_{1,n})$. The vertices $v_1, v_2, v_3, \ldots, v_n$ induces a complete graph and all the vertices in this complete graph are adjacent to $u$ corresponding to the centre vertex $u$ in the star graph. So $S = \{u\}$ is a power dominating set. Hence $\gamma_p(M(K_{1,n})) = 1$.

\section{Conclusion}

The power domination number $\gamma_p(G)$ is the minimum cardinality of a power dominating set of $G$. In this paper we have computed the power domination number of middle graphs of the path, cycle and star graph.

\section*{References}


