Bionomic equilibrium of a three species model with optimal harvesting for the first prey

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Abstract

This paper deals with the selective harvesting of a prey-predator system based on catch-per-unit-effort hypothesis. Here all the three species fulfill the logistic growth law. Using Routh-Hurwitz criteria the existence of positive equilibrium point and its local and global stability is studied. The extent of bionomic equilibrium has been established and the optimal harvesting policy is studied using Pontryagins maximum principle. Analysis of the effects of harvesting of one the prey species is worked out in this paper. The analytical results are illustrated using numerical examples.

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1 Introduction

A large amount of literature is available on the asymptotic stability and dynamical behavior of the Lotka-Volterra [1],[2] logistic models. Popular ecologists like Freedman[3], Kapur[4], Meyer[5], Cushing[6] have contributed extensively to this area and studied several models. An increased interest in the bionomic study of renewable resources like fisheries and forests has been observed in recent years. Natural resources exploitation has become a compassion all over the world. The bionomic exploitation and non-selective harvesting of independent population obeying law of logistic growth has been extensively studied by Clark[7],[8]. A two species prey-predator fishery model with harvesting and carrying capacities for both the species has been analyzed by authors like Tapasi Das[9]. In Recent years Papa Rao et al.[10], Shiva Reddy et al.[11] studied various interacting species models with optimal harvesting. Optimal harvesting of sustainable resources have been widely studied by authors like Kar[12], Chaudhuri[13], Ragozin and Brown[14].

In the present model, resembling a small fish, a star fish and a shark, and the shark predates on both the small fish and the star fish. And with the assumption of catch per unit effort criterion, we have taken up harvesting of the first prey species i.e., only the small fish is harvested for medicinal benefits and also as a resource of food. The interior equilibrium point of the proposed dynamical system is determined. The local and global stability of the interior equilibrium point is obtained and illustrated numerically. The occurrence of bionomic equilibrium and optimal harvesting policy has been discussed and explained in detail. Some concluding remarks based on the behavior of the system is made at the end.

Consider the following set of nonlinear ordinary differential equations.

\[
\frac{dN_1}{dt} = r_1N_1 \left( 1 - \frac{N_1}{L_1} \right) - \alpha N_1N_3 - \theta_1N_1 - qEN_1
\]

\[
\frac{dN_2}{dt} = r_2N_2 \left( 1 - \frac{N_2}{L_2} \right) - \beta N_2N_3 - \theta_2N_2
\]

\[
\frac{dN_3}{dt} = r_3N_3 \left( 1 - \frac{N_3}{L_3} \right) + \alpha_1 N_1N_3 + \beta_1 N_2N_3 - \theta_3N_3
\]
with the following notations: \(N_i\)'s are the population densities of all the three species. \(r_i\)'s are the intrinsic growth rates. \(\theta_i\)'s are the corresponding death rates and \(L_i\)'s are the carrying capacities of \(N_i\)'s (i=1, 2, 3). \(\alpha\) and \(\beta\) are the rates of decrease of \(N_1\) and \(N_2\) respectively. \(q\) is the catching capacity coefficient and \(E\) is the effort applied for harvesting the prey species \(N_1\).

2 The Interior Equilibrium of the System

\[
N_1 = \left[\frac{r_2r_3}{L_2L_3} + \beta\theta_1\right] \left(\frac{r_1 - \theta_1 - qE}{r_1} - \alpha\beta_1(r_2 - \theta_2) - \frac{\alpha r_2}{L_2}(r_3 - \theta_3)\right) - \frac{r_1}{L_1} \left[\frac{r_3}{L_2L_3} + \beta\theta_1\right] + \frac{\alpha r_1r_2}{L_2}
\]

(4)

\[
N_2 = \left[\frac{r_1r_3}{L_1L_3} + \alpha\theta_1\right] \left(\frac{r_2 - \theta_2}{r_2} - \alpha\beta(r_1 - \theta_1 - qE) - \frac{\beta r_1}{L_1}(r_3 - \theta_3)\right) - \frac{r_1}{L_1} \left[\frac{r_3}{L_2L_3} + \beta\theta_1\right] + \frac{\alpha r_1r_2}{L_2}
\]

(5)

\[
N_3 = \left[\frac{r_1r_2}{L_1L_2}\right] \left(\frac{r_1 - \theta_3}{r_1} + \frac{\alpha r_2}{L_2}(r_2 - \theta_2) + \frac{\alpha r_2(r_1 - \theta_1 - qE)}{L_2}\right) - \frac{r_1}{L_1} \left[\frac{r_3}{L_2L_3} + \beta\theta_1\right] + \frac{\alpha r_1r_2}{L_2}
\]

(6)

This state exists only when,

\[
\left[\frac{r_2r_3}{L_2L_3} + \beta\theta_1\right] (r_1 - \theta_1 - qE) > \alpha\beta_1(r_2 - \theta_2) + \frac{\alpha r_2}{L_2}(r_3 - \theta_3)
\]

\[
\left[\frac{r_1r_3}{L_1L_3} + \alpha\theta_1\right] (r_2 - \theta_2) > \alpha\beta(r_1 - \theta_1 - qE) + \frac{\beta r_1}{L_1}(r_3 - \theta_3)
\]

3 Local Stability analysis of the interior equilibrium point

Let \(N = (N_1, N_2, N_3)^T = \overline{N} + U\)

Where \(U = (u_1, u_2, u_3)^T\) is the small perturbation over the equilibrium state \(\overline{N} = (\overline{N}_1, \overline{N}_2, \overline{N}_3)^T\).
The equations of the perturbed state are obtained by linearizing the basic equations.

\[
\frac{dU}{dt} = AU
\]  

(7)

where

\[
A = \begin{bmatrix}
    r_1 - \frac{2r_1 N_1}{L_1} - \alpha N_3 & -\theta_1 - qE & 0 \\
    -\theta_1 - qE & r_2 - \frac{2r_2 N_2}{L_2} - \beta N_3 - \theta_2 & -\alpha N_1 \\
    \alpha_1 N_3 & \beta_1 N_3 & r_3 - \frac{2r_3 N_3}{L_3} + \alpha_1 N_1 + \beta_1 N_2 - \theta_3
\end{bmatrix}
\]

with the characteristic equation

\[
\lambda^3 + \left(\frac{r_1 N_1}{L_1} + \frac{r_2 N_2}{L_2} + \frac{r_3 N_3}{L_3}\right)\lambda^2
\]

\[
+ \left[\frac{r_1 r_2 N_1 N_2}{L_1 L_2} + \frac{r_2 r_3 N_2 N_3}{L_2 L_3} + \frac{r_1 r_3 N_1 N_3}{L_1 L_3} + \alpha_1 \overline{N_1 N_3} + \beta_1 \overline{N_2 N_3}\right] \lambda
\]

\[
+ \left(\frac{r_1 r_2 r_3}{L_1 L_2 L_3} + \frac{r_1 \beta_1}{L_1} + \frac{r_2 \alpha_1}{L_2}\right) \overline{N_1 N_2 N_3} = 0
\]  

(8)

Let

\[a_1 = \frac{r_1 N_1}{L_1} + \frac{r_2 N_2}{L_2} + \frac{r_3 N_3}{L_3}\]  

(9)

\[a_2 = \frac{r_1 r_2 N_1 N_2}{L_1 L_2} + \frac{r_2 r_3 N_2 N_3}{L_2 L_3} + \frac{r_1 r_3 N_1 N_3}{L_1 L_3} + \alpha_1 \overline{N_1 N_3} + \beta_1 \overline{N_2 N_3}\]  

(10)

\[a_3 = \left(\frac{r_1 r_2 r_3}{L_1 L_2 L_3} + \frac{r_1 \beta_1}{L_1} + \frac{r_2 \alpha_1}{L_2}\right) \overline{N_1 N_2 N_3}\]  

(11)

Using Routh-Hurwitz criteria, on simplification, we have, \(a_1 > 0\), and \(a_3(a_1 a_2 - a_3) > 0\)

Which proves that the interior equilibrium point is locally asymptotically stable.
4 Global stability analysis

Proof: Let \((N_1, N_2, N_3)\) be the interior equilibrium point. Consider the following Lyapunov function for the interior equilibrium point:

\[
V(N_1, N_2, N_3) = (N_1 - N_1) - N_1 \ln \left( \frac{N_1}{N_1} \right) + (N_2 - N_2) - N_2 \ln \left( \frac{N_2}{N_2} \right) + (N_3 - N_3) - N_3 \ln \left( \frac{N_3}{N_3} \right) \tag{12}
\]

Then time derivative along \(V\) is obtained as follows

\[
\frac{dV}{dt} = \frac{dN_1}{dt} \left[ 1 - \frac{N_1}{N_1} \right] + \frac{dN_2}{dt} \left[ 1 - \frac{N_2}{N_2} \right] + \frac{dN_3}{dt} \left[ 1 - \frac{N_3}{N_3} \right] =
\]

\[
(N_1 - N_1) \left[ r_1 - \frac{r_1 N_1}{L_1} - \alpha N_3 - \theta_1 - qE \right] + (N_2 - N_2) \left[ r_2 - \frac{r_2 N_2}{L_2} - \beta N_3 - \theta_2 \right] + (N_3 - N_3) \left[ r_3 - \frac{r_3 N_3}{L_3} + \alpha_1 N_1 + \beta_1 N_2 - \theta_3 \right] \tag{13}
\]

Substituting,

\[
r_1 - \theta_1 - qE = \frac{r_1}{L_1} N_1 + \alpha N_3
\]

\[
r_2 - \theta_2 = \frac{r_2}{L_2} N_2 + \beta N_3
\]

\[
r_3 - \theta_3 = \frac{r_3}{L_3} N_3 - \alpha_1 N_1 - \beta_1 N_2
\]

we get

\[
\frac{dV}{dt} = -\frac{r_1}{L_1} \left[ N_1 - N_1 \right] - \frac{r_2}{L_2} \left[ N_2 - N_2 \right] - \frac{r_3}{L_3} \left[ N_3 - N_3 \right] - (\alpha - \alpha_1) [N_1 - N_1] [N_3 - N_3] - (\beta - \beta_1) [N_2 - N_2] [N_3 - N_3]
\]

\[
\leq - \left[ \frac{r_1}{L_1} + \frac{(\alpha - \alpha_1)}{2} \right] \left[ N_1 - N_1 \right] - \left[ \frac{r_2}{L_2} + \frac{(\beta - \beta_1)}{2} \right] \left[ N_2 - N_2 \right] - \left[ \frac{r_3}{L_3} + \frac{(\alpha - \alpha_1 + \beta - \beta_1)}{2} \right] \left[ N_3 - N_3 \right] < 0 \tag{15}
\]

Hence the system is globally asymptotically stable.
5 Bionomic Equilibrium

The term Bionomic Equilibrium is a conjunction of both biological equilibrium and economic equilibrium. As we have already seen, the biological equilibrium is given by

\[ \frac{dN_i}{dt} = 0, \ i = 1, 2, 3. \] (16)

and the economic equilibrium is said to be achieved when TR (the total revenue obtained by selling the harvested biomass) equals TC (the total cost for the effort devoted to harvesting).

Let, \( c \) be the cost of harvesting per unit effort, \( p \) be the price per unit biomass of the prey. The economic rent is given by

\[ R = (pqN_1 - c)E. \]

The bionomic equilibrium \((N_1)_\infty, (N_2)_\infty, (N_3)_\infty, (E)_\infty\) is given by the following equations:

\[
\begin{align*}
    r_1 N_1 \left[ 1 - \frac{N_1}{L_1} \right] - \alpha N_1 N_3 - \theta_1 N_1 - qEN_1 &= 0 \\
    r_2 N_2 \left[ 1 - \frac{N_2}{L_2} \right] - \beta N_2 N_3 - \theta_2 N_2 &= 0 \\
    r_3 N_3 \left[ 1 - \frac{N_3}{L_3} \right] + \alpha_1 N_1 N_3 + \beta_1 N_2 N_3 - \theta_3 N_3 &= 0
\end{align*}
\]

\[ R = (pqN_1 - c)E. \]

Consider the following cases to determine the bionomic equilibrium:

**Case(i):** When \( c > pqN_1 \), the cost becomes more than the revenue for prey species and hence the prey species cannot be harvested.

**Case(ii):** When \( c < pqN_1 \), the cost will be less than the revenue for prey species and hence it can be harvested. Therefore,

\[
\begin{align*}
    (N_1)_\infty &= \frac{c}{pq}, \\
    (N_2)_\infty &= \frac{\frac{r_2}{L_2} (r_2 - \theta_2) - \beta \left( r_3 - \theta_3 + \alpha_1 \frac{c}{pq} \right)}{\frac{r_2 r_3}{L_2 L_3} + \beta \beta_1}.
\end{align*}
\]
\[(N_3)_\infty = \frac{\beta_1(r_2 - \theta_2) + \frac{r_2}{L_2} \left( r_3 - \theta_3 + \alpha_1 \frac{c}{pq} \right)}{\frac{r_2}{L_2} + \beta_\beta_1}, \]

Also,
\[(E)_\infty = \frac{1}{q} \left[ r_1 - \frac{r_1 c}{L_1pq} - \alpha(N_3)_\infty - \theta_1 \right] \]

And,
\[(E)_\infty > 0, \]

provided if
\[r_1 > \frac{r_1 c}{L_1pq} + \alpha(N_3)_\infty + \theta_1 \]

Thus, the bionomic equilibrium point exists if the above condition is true.

6 Optimal Harvesting

In this section, our objective is to maximize the present value \(J\) of
continuous time stream of revenues defined by
\[J = \int_0^\infty e^{-\delta t} \left[ (pqN_1 - c)E(t) \right] dt \]

where \(\delta\) denotes the instantaneous annual rate of discount and we have to maximize \(J\) subject to the state equations (1-3) by Pontryagins maximum principle[15].The control variable \(E(t)\) is subjected to the constraint \(0 \leq E(t) \leq (E)_{max}\). The Hamiltonian for the problem is given by
\[
H = e^{-\delta t} \left[ (pqN_1 - c)E \right] + \lambda_1 \left[ r_1 N_1 - \frac{r_1}{L_1} N_1^2 - \alpha N_1 N_3 - \theta_1 N_1 - qEN_1 \right] \\
+ \lambda_2 \left[ r_2 N_2 - \frac{r_2}{L_2} N_2^2 - \beta N_2 N_3 - \theta_2 N_2 \right] \\
+ \lambda_3 \left[ r_3 N_3 - \frac{r_3}{L_3} N_3^2 + \alpha N_1 N_3 + \beta N_2 N_3 - \theta_3 N_3 \right] \quad (17)
\]
Where \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) are called the adjoint variables and the control variable \( E \) appears linear in the Hamiltonian. By Pontryagins maximum principle,

\[
\frac{\partial H}{\partial E} = 0; \quad \frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial N_1}; \quad \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial N_2}; \quad \frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial N_3};
\]  

(18)

Now

\[
\frac{\partial H}{\partial E} = 0; \quad \Rightarrow e^{-\delta t}(pqN_1 - c) - \lambda_1 q N_1 = 0
\]

(19)

\[
\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial N_1} \Rightarrow \frac{d\lambda_1}{dt} = -e^{-\delta t}pqE + \frac{\lambda_1 r_1 N_1}{L_1} - \lambda_3 \alpha_1 N_3
\]

(20)

\[
\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial N_2} \Rightarrow \frac{d\lambda_2}{dt} = \frac{\lambda_2 r_2 N_2}{L_2} - \lambda_3 \beta_1 N_3
\]

(21)

\[
\frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial N_3} \Rightarrow \frac{d\lambda_3}{dt} = \lambda_1 \alpha N_1 + \lambda_2 \beta N_2 + \frac{\lambda_3 r_3 N_3}{L_3}
\]

(22)

From (19) and (22) we have

\[
\frac{d\lambda_3}{dt} = \alpha e^{-\delta t} \left( pN_1 - \frac{c}{q} \right) + \lambda_2 \beta N_2 + \lambda_2 \beta N_2 + \frac{\lambda_3 r_3 N_3}{L_3}
\]

(23)

Solving for \( \lambda_2 \) & \( \lambda_3 \) from (21) and (23) we get,

\[
\lambda_2 = A_1 e^{s_1 t} + A_2 e^{s_2 t} + Be^{-\delta t}
\]

(24)

\[
\lambda_3 = A_3 e^{s_1 t} + A_4 e^{s_2 t} + De^{-\delta t}
\]

(25)

Where \( s_1 \) & \( s_2 \) are the roots of the equations (24) & (25) and

\[
A_1 = \frac{\left( s_1 - \frac{r_3 N_3}{L_3} \right) \left( s_1 + \delta \right) \lambda_20 - \beta_1 N_3 \left[ \alpha \left( pN_1 - \frac{c}{q} \right) + \lambda_30 \left( s_1 + \delta \right) \right]}{(s_1 - s_2)(s_1 + \delta)}
\]

\[
A_2 = \frac{\left( s_2 - \frac{r_3 N_3}{L_3} \right) \left( s_2 + \delta \right) \lambda_20 - \beta_1 N_3 \left[ \alpha \left( pN_1 - \frac{c}{q} \right) + \lambda_30 \left( s_2 + \delta \right) \right]}{(s_2 - s_1)(s_2 + \delta)}
\]

\[
B = \frac{-\beta_1 N_3 \alpha \left( pN_1 - \frac{c}{q} \right)}{(s_1 + \delta)(s_2 + \delta)}
\]
\[ A_3 = \left( s_1 - \frac{r_2 N_2}{L_2} \right) \left[ \alpha \left( p N_1 - \frac{c}{q} \right) + \lambda_{30} (s_1 + \delta) \right] + \beta N_2 \lambda_{20} (s_1 + \delta) \]

\[ A_4 = \left( s_2 - \frac{r_2 N_2}{L_2} \right) \left[ \alpha \left( p N_1 - \frac{c}{q} \right) + \lambda_{30} (s_2 + \delta) \right] + \beta N_2 \lambda_{20} (s_2 + \delta) \]

\[ D = -\left( \delta + \frac{r_2 N_2}{L_2} \right) \left( p N_1 - \frac{c}{q} \right) \alpha \]

Where \( \lambda_{20} \) & \( \lambda_{30} \) represents the initial strengths of \( \lambda_2 \) & \( \lambda_3 \) respectively.

Now substituting \( \lambda_3 \) in (20),

\[ \frac{d\lambda_1}{dt} - \frac{r_1 N_1}{L_1} \lambda_1 = e^{-\delta t} pq E - \alpha_1 N_3 (A_3 e^{s_1 t} + A_4 e^{s_2 t} + D e^{-\delta t}) \]

which becomes linear in \( \lambda_1 \) and its complete solution is given by,

\[ \lambda_1 = pq E \left( \frac{1 + \alpha_1 N_3 D}{\delta + \frac{r_1 N_1}{L_1}} \right) e^{-\delta t} - \frac{\alpha_1 N_3 A_3}{s_1 - \frac{r_1 N_1}{L_1}} e^{s_1 t} - \frac{\alpha_1 N_3 A_3}{s_2 - \frac{r_1 N_1}{L_1}} e^{s_2 t} \]

(26)

It is clear from (26) that \( \lambda_1 \) is bounded if and only if \( s_i < 0 \) (i=1,2) or \( A_i \) (i=1,2,3,4) are identically zero. For well made calculations we ignore the cases where \( s_i < 0 \) and take \( A_i \equiv 0 \).

Therefore, we have

\[ \lambda_1(t) = pq E \left( \frac{1 + \alpha_1 N_3 D}{\delta + \frac{r_1 N_1}{L_1}} \right) e^{-\delta t} \]

(27)

Similarly,

\[ \lambda_2 = Be^{-\delta t} \]

(28)

\[ \lambda_3 = De^{-\delta t} \]

(29)

Hence from (19) and (27), we get a singular path,

\[ e^{-\delta t} \left( p - \frac{c}{q N_1} \right) = pq E \left( \frac{1 + \alpha_1 N_3 D}{\delta + \frac{r_1 N_1}{L_1}} \right) e^{-\delta t} \]
Which shows that there exists a unique positive root $N_1^* = (N_1)\delta$ in the interval $0 < N_1^* < P_1$. From this we have,

\[
N_1(\delta) = \frac{c}{pq},
\]

\[
N_2(\delta) = \frac{c_p (r_2 - \theta_2) - \beta (r_3 - \theta_3 + \alpha N_1(\delta))}{\frac{r_p}{L_3}} + \beta_1 \frac{r_2}{L_2 L_3}, \quad (30)
\]

\[
N_3(\delta) = \frac{\beta_1 (r_2 - \theta_2) + \frac{c_p}{L_2} (r_3 - \theta_3 + \alpha_1 N_1(\delta))}{\frac{r_2}{L_2 L_3} + \beta_1},
\]

Also,

\[
(E)(\delta) = \frac{1}{q} \left[ r_1 - \frac{r_1}{N_1(\delta)} - \alpha (N_3(\delta) - \theta_1) \right], \quad (31)
\]

Hence an optimal equilibrium point is determined and the optimal harvesting effort is established. We note that $e^{\delta t}\lambda_i(t), \ (i = 1, 2, 3)$, remains constant in an optimal equilibrium and hence satisfies the transversality condition at $\infty$, i.e., they remain bounded as $t \to \infty$.

7 Numerical Simulations

Case (i): $r_1=2.5; \ r_2=1.5; \ r_3=0.5; \ L_1=10; \ L_2=20; \ L_3=10; \ \alpha=0.05; \ \alpha_1=0.01; \ \beta=0.05; \ \beta_1=0.01; \ \theta_1=0.2; \ \theta_2=0.2; \ \theta_3=0.2; \ q=0.01; \ \text{From figures 1a, 1b, 1c & 1d, we observe the variations in all the three species with respect to time t, when their initial strengths are assumed as 10, 12 & 15 with q=0.01 and E=10 & 50 respectively. Comparing both the figures we observe that the population of the first prey species falls down quickly and stabilizes after some time period. Also it can be analyzed that increase in the effort applied for harvesting the first prey species affects the population of the predator species. The phase portrait also shows that when the time increases the population tends to an equilibrium point.}$

Case (ii): $r_1=2.5; \ r_2=1.5; \ r_3=0.5; \ L_1=10; \ L_2=20; \ L_3=10; \ \alpha_1=0.01; \ \beta=0.05; \ \beta_1=0.01; \ \theta_1=0.2; \ \theta_2=0.2; \ \theta_3=0.2; \ q=0.01; \ E=50; \ \text{Figures 2a, 2b, 2c & 2d shows the variations again in all the three species with respect to time when the interacting coefficient \alpha is 0.1 and 0.2 respectively. With the catching capacity coefficient as 0.01 and the effort applied for harvesting the first prey species as 50, the}$
phase space portrait shows that the system reaches the equilibrium state after some time period and becomes globally asymptotically
stable for the assumed parameter values.

8 Conclusions

The paper proposes a three species ecological model with two preys and one predator. The catch-per-unit-effort criterion is considered to distinguish this from other models. The first prey is subjected to harvesting continuously with minimum economic profits. The local and global stability of the system has been established and the existence of bioeconomic equilibrium is examined. The optimal harvesting policy has been studied using Pontryagins maximum principle. At the steady state condition, it has been observed that the population of the first prey can be maintained at an equilibrium level and the harvesting cost equals the marginal profit per unit effort. The stability of the system is well explained with the help of numerical simulation and the graphs using MATLAB by taking suitable values to the parameters satisfying the assumptions.

References


