

Optimal Control Strategy for a Discrete Model on Depletion of Water Resources

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February 10, 2017

Abstract

In this paper, we construct a discrete time model on depletion of water resources. We investigate the optimal control strategy of the model. With an aim to reduce the quantity of the water resources being depleted, we use a control simulating an awareness program. Pontryagins maximum principle, in discrete time, is used to characterize the optimal control. The numerical simulation is carried out using MATLAB. The obtained results confirm the performance of the optimization strategy.

AMS Subject Classification: 39A10, 34H05, 93A30.

Key Words and Phrases: Difference equations, Optimal Control, Pontryagins Maximum Principle, Hamiltonian.

1 Introduction

Water is the most important element of the biosphere which is essential for the existence of all forms of life. In a civilized society, the

human activities and economic development are dependent upon water. 70 percent of the earth surface is covered with water. However, 97.5 percent of this water being sea water, it is salty. Fresh water availability is only 35 m km³. Out of the total fresh water, 68.7 percent is frozen in ice caps, 30 percent is stored underground and only 0.3 percent water is available on the surface of the earth. Out of the surface water, 87 percent is stored in lakes, 11 percent in swamp and 2 percent in rivers. Only 1 percent of the total water can be used by human beings. The freshwater bodies may be lotic (i.e., running water as spring, stream, rivers, etc.). As water was available in plenty, it was considered as a free resource since generations. However, with growing demand for water and depletion of the available water, assured supply of good quality water is becoming a growing concern.

India receives good rainfall well distributed over 5-6 months in the year. The average annual rainfall in the country is 1170 mm with a wide range between 100 mm in desert areas of Rajasthan to 10000 mm in Cherapunji. It is disturbing to note that only 18 percent of the rainwater is used effectively while 48 percent enters the river and most of which reaches the ocean. Out of the total usable water, 728 billion m³ is contributed from surface water and 395 billion m³ is contributed by replenishable ground water. Against the above supply, the water consumed during the year 2006 in India was 829 billion m³ which is likely to increase to 1093 billion m³ in 2025 and 1047 billion m³ in 2050, as estimated by the Government of India (2009). As the potential for increasing the volume of utilisation of water is hardly 5-10 percent, India is bound to face severe scarcity of water in the near future.

Optimal control theory, an extension of the calculus of variations, is a mathematical optimization method for deriving control policies. The method is largely due to the work of Lev Pontryagin and Richard Bellman in the 1950s, after contributions to calculus of variations by Edward J. McShane. An optimal control problem is an optimization problem. Optimal control deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved. A control problem includes a cost functional that is a function of state and control variables. An optimal control is a set of differential equations describing the paths of the control variables that minimize the cost function. The optimal

control can be derived using Pontryagin's maximum principle (a necessary condition also known as Pontryagin's minimum principle or simply Pontryagin's Principle), or by solving the Hamilton-Jacobi-Bellman equation (a sufficient condition).

In this paper, we construct a mathematical model on depletion of water resources and introduce the control of awareness measures. In section 2, a mathematical model is proposed. In section 3, we investigate the optimal control problem. Section 4 consists of Numerical simulations through MATLAB. Conclusion is given in section 5.

2 The Mathematical Model

In this paper, we consider that the human population are dependent on water resources for their livelihood. And hence it is being depleted by human population. We construct a discrete time mathematical model on depletion of water resources using a system of difference equations:

$$\begin{aligned} X_{n+1} &= X_n + X_n(1 - X_n) - a_1 X_n Y_n - a_2 X_n \\ Y_{n+1} &= Y_n + Y_n(1 - Y_n) + b_1 X_n Y_n + (b_2 - b_3) Y_n \end{aligned} \quad (1)$$

Where

X_n -Averaged discharge in a section of river(Water resources) at time n .

Y_n -Density of the human population at time n .

a_1 - Rate of Depletion of the Water resources due to human population.

a_2 - Natural depletion rate of the Water resources.

b_1 - Rate of usage of the Water resources by human population.

b_2 - Growth rate of human population due to the Water resources.

b_3 - Death rate of human population.

The strategy of control we adopt consists of an awareness program. Our goal is to minimize the depletion of water resources. In this model we include a control u_n , that represents the awareness measures at time n . So the controlled mathematical system is given by the following system of difference equations.

$$\begin{aligned}X_{n+1} &= X_n + X_n(1 - X_n) - a_1 X_n Y_n - a_2 X_n - u_n X_n \\Y_{n+1} &= Y_n + Y_n(1 - Y_n) + b_1 X_n Y_n + (b_2 - b_3) Y_n\end{aligned}\quad (2)$$

3 The Optimal Control Problem

In this section, we use the optimal control theory to analyze the behavior of the model. Our aim is to reduce the depletion of water resources through public awareness during the time steps $n = 0$ to T and also minimizing the cost spent in the public awareness programs. We are assuming the cost of administering the control is quadratic for simplicity. The problem is to minimize the objective functional (see [2],[3],[5]).

$$J(u) = A_T X_T + \sum_{n=0}^T \left(A_n X_n + \frac{B_n}{2} u_n^2 \right) \quad (3)$$

Where the parameters $A_n > 0$ and $B_n > 0$ are the cost coefficients they are selected to weigh the relative importance of X_n and u_n at time n . T is the final time. We are minimizing the quantity of water resources being depleted during the time steps $n = 0$ to $T - 1$, and at the final time and also minimizing the cost of administering the control.

In other words, we seek the optimal control u^* such that

$$J(u^*) = \min J(u) \quad (4)$$

Where U is the set of admissible controls defined by

$$U = \{u_n : a \leq u_n \leq b, n = 0, 1, 2, \dots, T - 1\} \quad (5)$$

In order to derive the necessary condition for optimal control, the pontryagins maximum principle, in discrete time, given in was used. This principle converts into a problem of minimizing a Hamiltonian, H_n at time step n defined by

$$H_n = A_n X_n + \frac{B_n}{2} u_n^2 + \sum_{j=1}^2 \lambda_{j,n+1} f_{j,n+1} \quad (6)$$

where $f_{j,n+1}$ is the right side of the difference equation of the j^{th} state variable at time step $n + 1$.

Theorem 1. Given an optimal control $u_n^* \in U$ and the solutions X_n^* and Y_n^* of the corresponding state system (2), there exists adjoint functions $\lambda_{1,n}$ and $\lambda_{2,n}$ satisfying

$$\lambda_{1,n} = A_n + \{2(1 - X_n) - a_1 Y_n - a_2 - u_n\} \lambda_{1,n+1} + b_1 Y_n \lambda_{2,n+1} \quad (7)$$

$$\lambda_{2,n} = \{2(1 - Y_n) + b_1 X_n + (b_2 - b_3)\} \lambda_{2,n+1} - a_1 X_n \lambda_{1,n+1} \quad (8)$$

With the transversality conditions at time T $\lambda_{1,T} = A_T$ and $\lambda_{2,T} = 0$. Furthermore, for $n = 0, 1, 2, \dots, T-1$, the optimal control u_n^* is given by

$$u_n^* = \min \left(b, \max \left(a, \frac{\lambda_{1,n+1}}{B_n} X_n \right) \right) \quad (9)$$

Proof. The Hamiltonian at time step n is given by

$$H_n = A_n X_n + \frac{B_n}{2} u_n^2 + \lambda_{1,n} \{X_n + X_n(1 - X_n) - a_1 X_n Y_n - a_2 X_n - u_n X_n\} \\ + \lambda_{2,n} \{Y_n + Y_n(1 - Y_n) + b_1 X_n Y_n + (b_2 - b_3) Y_n\} \quad (10)$$

$$\lambda_{1,n} = \frac{\partial H_n}{\partial X_n}, \lambda_{1,T} = A_T \quad (11)$$

$$\lambda_{2,n} = \frac{\partial H_n}{\partial Y_n}, \lambda_{2,T} = 0 \quad (12)$$

For, $n = 0, 1, \dots, T-1$ the optimal control u_n^* can be solved from the optimality condition,

$$\frac{\partial H_n}{\partial u_n} = 0$$

That is, $\frac{\partial H_n}{\partial u_n} = B_n u_n - \lambda_{1,n+1} X_n = 0$ □

4 Numerical Simulation

We carry out numerical simulations using MATLAB. The graphs given below allows us to compare changes in the density of average discharge depleted and the rate of usage of discharge by human population before and after the introduction of control.

Figure 2 shows the effect of control in reducing the depletion of average discharge.

Figure 3 shows that, the rate of usage of the discharge by human population decrease rapidly when the control is introduced.

Figure 4 gives a representation of optimal control u_n^* .

S. No	Parameter	Value
1.	a_1	0.5
2.	a_2	0.12
3.	b_1	0.417
4.	b_2	0.65
5.	b_3	0.5

Figure 1: Values of the Parameters

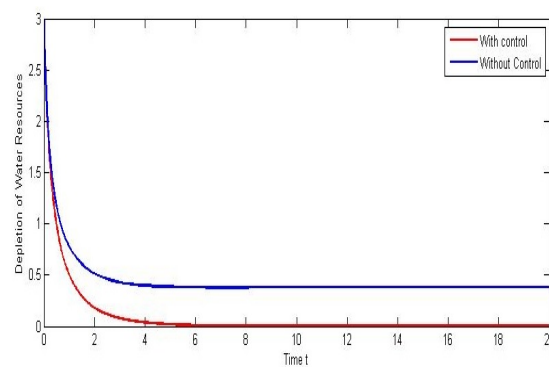


Figure 2: Depletion of water resources with and without control

5 Conclusion

In this paper, we have applied optimal control techniques to Ecology. The optimal control theory has been applied to model depletion on water resources, where a control representing the percentage of depletion of water resources being provided awareness per unit. Numerical simulations through MATLAB show that the proposed control strategy is efficient in decreasing the rate of usage of water resources by human population and the cost of awareness programs.

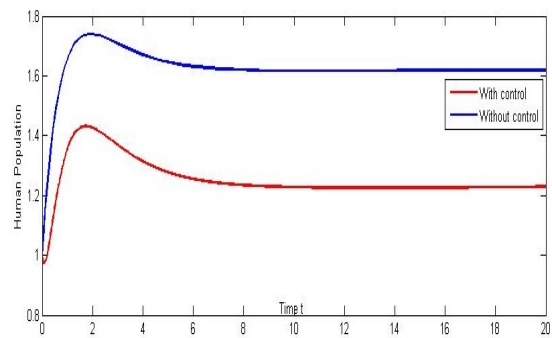


Figure 3: Rate of usage of the discharge by human population with and without control

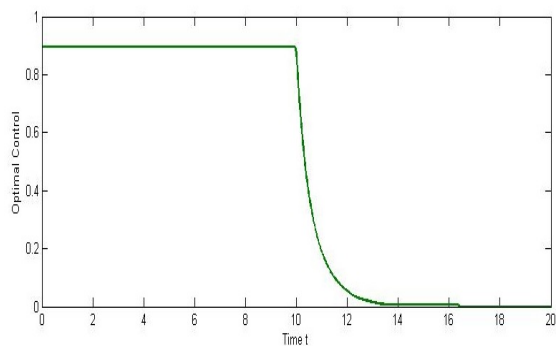


Figure 4: The optimal control u_n

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