

EFFECT OF MULTIPLE STENOSES ON FLOW OF A JEFFREY FLUID THROUGH A POROUS MEDIUM

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ABSTRACT

In this paper, an attempt has been made to analyse the flow of a Jeffrey fluid in a stenosed channel filled with porous material. Assuming the stenoses to be mild, the governing equations of the flow have been solved and closed form expressions for the resistive impedance and the wall shear stress have been determined. The effects of different relevant parameters on these flow characteristics have been investigated. The resistance to flow and the wall shear stress increase with the heights of the stenoses and permeability but decreases with Jeffrey parameter. Moreover, the wall shear stress decreases with Jeffrey parameter and Darcy number and tends to increase with the heights of stenoses.

1. Introduction

Stenosis is the term used for the abnormal and unnatural narrowing in a blood vessel or other tubular organs or structures. The deposition of lipid and cholesterol particles and proliferation of convective tissues in an artery wall form plaques, which grow inside and restrict the blood flow. This is the main reason which causes stenosis. Since atherosclerosis is one of the most wide spread diseases, it is necessary to understand it as much as possible.

In view of this, flow through stenosed tubes has received considerable attention in the recent past. Young [1], Forrester and Young [2] and Chaturani and Ponnalagarsamy [3] considered the flow of blood through an axisymmetric tube with mild stenosis under different conditions. However, in all these studies blood has been characterized as a Newtonian fluid and little attention has been given to its suspension nature. Some experimental studies indicate that blood exhibits non-Newtonian behavior at low shear rates in tubes of small diameters (Huckaba and Hahn [4]). Hence, several authors have studied blood flow in stenosed region considering blood as a non-Newtonian fluid (Shukla et al. [5], Mekhiemer and Kot [6] and Vajravelu et al. [7]).

The non-Newtonian fluids, exhibiting a non-linear relationship between the stresses and the rate of strain, are now acknowledged as more appropriate models in medical sciences and technological applications than Newtonian fluids. The simplest subclass of rate type fluids is the Jeffrey's version of the Oldroyd model (Oldroyd [8]), in which the relaxation phenomena are taken

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into consideration. This fluid model includes derivatives of the rate of strain tensor, besides time derivatives of the stress tensor, to show relaxation as well as retardation behavior. Further, Jeffrey fluid model includes elastic and memory effects exhibited by dilute polymer solutions and biological liquids (Hayat et al. [9]). Because of its physiological relevance, Kothandapani and Srinivas [10], Pandey and Tripathi [11] and Srinivas et al. [12] studied the transport of Jeffrey fluid by considering various situations.

2. MATHEMATICAL MODEL

Consider a steady and incompressible Jeffrey fluid flow through a uniform channel of width $2d_0$ with multiple stenoses. It is assumed that the channel is filled with porous material. A Cartesian coordinate system (x, y) is chosen so that the x -axis coincides with the center line of the channel and the y -axis normal to it. Further, the stenoses are supposed to be mild and develop in a symmetric manner. The geometry of the wall is taken as (Fig. 1)

$$(1) \quad \eta(x) = \begin{cases} d_0 & : 0 \leq x \leq d_1 \\ d_0 - \frac{\delta_1}{2} [1 + \cos \frac{2\pi}{L_1}(x - d_1 - \frac{L_1}{2})] & : d_1 \leq x \leq d_1 + L_1 \\ d_0 & : d_1 + L_1 \leq x \leq d_2 \\ d_0 - \frac{\delta_2}{2} [1 + \cos \frac{2\pi}{L_2}(x - d_2 - \frac{L_2}{2})] & : d_2 \leq x \leq d_2 + L_2 \\ d_0 & : d_2 + L_2 \leq x \leq L \end{cases}$$

where L is the length of the channel, L_i and δ_i ($i = 1, 2$) are the lengths and maximum heights of two mild stenoses (Fig. 1).

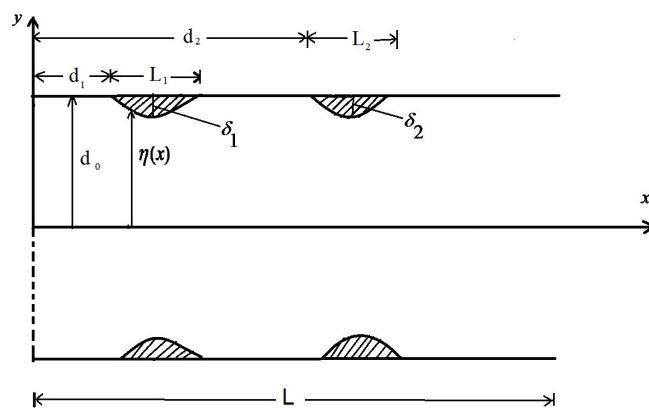


FIGURE 1. Geometry of the problem

The equations of motion characterizing an incompressible Jeffrey fluid flow, (Hayat et al. [9]), are given by

$$(2) \quad \nabla \cdot \bar{q} = 0,$$

$$(3) \quad \rho \frac{D\bar{q}}{Dt} = -\nabla p + \nabla \cdot \bar{S},$$

where $\bar{q}$ is the velocity vector, p is the pressure, ρ is the density, t is the time and $\bar{\mathbf{S}}$ is the extra stress tensor.

The equations (2) and (3) respectively represent the principles of conservation of mass and linear momentum.

The constitutive equations for the Jeffrey fluid are

$$(4) \quad \bar{\mathbf{T}} = -p\bar{\mathbf{I}} + \bar{\mathbf{S}}$$

$$(5) \quad \bar{\mathbf{S}} = \frac{\mu}{1 + \lambda_1}(\dot{\gamma} + \lambda_2 \ddot{\gamma})$$

where $\bar{\mathbf{T}}$ is the Cauchy stress tensor, $\bar{\mathbf{I}}$ is the identity tensor, λ_1 is the ratio of relaxation to retardation times, λ_2 is the retardation time, μ is the dynamic viscosity, $\dot{\gamma}$ is the shear rate and dots over the quantities indicate differentiation with respect to time.

The equations governing the two dimensional motion of an incompressible Jeffrey fluid for the present problem are given by

$$(6) \quad \rho \left[\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right] u = -\frac{\partial p}{\partial x} + \frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial y} - \frac{\mu}{k_0} u$$

$$(7) \quad \rho \left[\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right] v = -\frac{\partial p}{\partial y} + \frac{\partial S_{yy}}{\partial y} + \frac{\partial S_{yx}}{\partial x} - \frac{\mu}{k_0} v$$

and the equation of continuity is

$$(8) \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

where u, v are the velocity components in the x and y directions respectively, p is the pressure, ρ is the density, $S_{xx}, S_{xy}, S_{yx}, S_{yy}$ are extra stress components and μ is the viscosity coefficient and k_0 is the permeability of the medium.

Assuming the flow to be steady and the stenosis to be mild (Young [1]), the governing equations for the present problem reduce to,

$$(9) \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$(10) \quad \frac{\mu}{1 + \lambda_1} \frac{\partial^2 u}{\partial y^2} + \frac{\mu}{k_0} u = \frac{\partial p}{\partial x}$$

$$(11) \quad \frac{\partial p}{\partial y} = 0$$

The no-slip boundary conditions are given by

$$(12) \quad u = 0 \quad \text{at} \quad y = \pm \eta(x)$$

Solving (9)-(11) under the boundary condition (12), the velocity is given as

$$(13) \quad u(y) = \frac{k_0}{\mu} \frac{\partial p}{\partial x} \left[\frac{\cosh(b_1 y)}{\cosh(b_1 \eta)} - 1 \right]$$

$$(14) \quad \text{where } b_1 = \sqrt{\frac{1 + \lambda_1}{k_o}}$$

3. ANALYSIS

The flux 'Q' of the fluid is given by

$$(15) \quad Q = 2 \int_0^\eta u \, dy = \frac{k_o}{\mu} \frac{\partial p}{\partial x} [\tanh(b_1 \eta) - b_1 \eta]$$

Introducing the following dimensionless quantities,

$$(16) \quad \delta'_1 = \frac{\delta_1}{d_o}; \quad (x', d'_1, L'_1, L'_2) = \frac{(x, d_1, L_1, L_2)}{L}; \quad \delta'_2 = \frac{\delta_2}{d_o}; \quad H = \frac{\eta}{d_o};$$

in equation (1) and equation (15) we get (after dropping primes)

$$(17) \quad Q = \frac{2d_o^3 Da}{\mu} \sqrt{\frac{Da}{1 + \lambda_1}} \frac{\partial p}{\partial x} \left[\tanh\left(\sqrt{\frac{1 + \lambda_1}{Da}} H\right) - H \sqrt{\frac{1 + \lambda_1}{Da}} \right]$$

$$(18) \quad \text{where } Da = \frac{k_o}{d_o^2}$$

From equation (17), we get

$$(19) \quad \frac{\partial p}{\partial x} = \frac{\mu Q}{2d_o^3 Da} \sqrt{\frac{1 + \lambda_1}{Da}} \left[\frac{1}{\tanh\left(\sqrt{\frac{1 + \lambda_1}{Da}} H\right) - H \sqrt{\frac{1 + \lambda_1}{Da}}} \right]$$

Integrating equation (19) with respect to x , we get pressure difference Δp along the total length of the channel as:

$$(20) \quad \Delta p = \frac{\mu Q}{2d_o^3 Da} \sqrt{\frac{1 + \lambda_1}{Da}} \int_0^1 \left[\frac{1}{\tanh\left(\sqrt{\frac{1 + \lambda_1}{Da}} H\right) - H \sqrt{\frac{1 + \lambda_1}{Da}}} \right] dx$$

The resistance to the flow, denoted by λ , is defined by

$$(21) \quad \lambda = \frac{\Delta p}{Q}$$

Using equation (20) in equation (21), we get

$$(22) \quad \lambda = \frac{\mu}{2d_o^3 Da} \sqrt{\frac{1 + \lambda_1}{Da}} \int_0^1 \left[\frac{1}{\tanh\left(\sqrt{\frac{1 + \lambda_1}{Da}} H\right) - H \sqrt{\frac{1 + \lambda_1}{Da}}} \right] dx$$

The pressure drop in the case of no stenosis ($H = 1$), denoted by Δp_n , is obtained from equation (20), as

$$(23) \quad \Delta p_n = \frac{\mu Q}{2d_o^3 Da} \sqrt{\frac{1 + \lambda_1}{Da}} \int_0^1 \left[\frac{1}{\tanh\left(\sqrt{\frac{1 + \lambda_1}{Da}}\right) - \sqrt{\frac{1 + \lambda_1}{Da}}} \right] dx$$

The resistance to the flow in the absence of stenosis ($H = 1$), denoted by λ_n , is obtained from (22), as

$$(24) \quad \frac{\Delta p_n}{Q} = \lambda_n = \frac{\mu}{2d_o^3 Da} \sqrt{\frac{1+\lambda_1}{Da}} \int_0^1 \left[\frac{1}{\tanh\left(\sqrt{\frac{1+\lambda_1}{Da}}\right) - \sqrt{\frac{1+\lambda_1}{Da}}} \right] dx$$

The normalized resistance to the flow, $\bar{\lambda}$ is given by

$$(25) \quad \bar{\lambda} = \frac{\lambda}{\lambda_n} = \left(\tanh\left(\sqrt{\frac{1+\lambda_1}{Da}}\right) - \sqrt{\frac{1+\lambda_1}{Da}} \right) \int_0^1 \left[\frac{1}{\tanh\left(\sqrt{\frac{1+\lambda_1}{Da}} H\right) - H \sqrt{\frac{1+\lambda_1}{Da}}} \right] dx$$

The shear stress acting on the wall of the channel is given by

$$(26) \quad \tau_w = -\mu \frac{\partial u}{\partial y} \Big|_{y=\eta}$$

Using equation (13), we get

$$(27) \quad \tau_w = k_o \frac{\mu Q}{2d_o^3 Da} \sqrt{\frac{1+\lambda_1}{Da}} \left[\frac{1}{\tanh\left(\sqrt{\frac{1+\lambda_1}{Da}} H\right) - H \sqrt{\frac{1+\lambda_1}{Da}}} \right] b_1 \tanh(b_1 \eta)$$

Introducing dimensionless quantity

$$(28) \quad \tau'_w = \frac{\tau_w}{\left(\frac{\mu U}{d_o}\right)}$$

Then from equation (27), we get,

$$(29) \quad \tau'_w = \frac{Q}{2U d_o} \left(\frac{1+\lambda_1}{Da} \right) \left[\frac{1}{\tanh\left(\sqrt{\frac{1+\lambda_1}{Da}} H\right) - \sqrt{\frac{1+\lambda_1}{Da}} H} \right] \tanh\left(\sqrt{\frac{1+\lambda_1}{Da}} H\right)$$

The shear stress at the wall in the absence of stenosis ($H=1$), denoted by $(\tau'_w)_n$, can be obtained from equation (29) as

$$(30) \quad (\tau'_w)_n = \frac{Q}{2U d_o} \left(\frac{1+\lambda_1}{Da} \right) \left[\frac{1}{\tanh\left(\sqrt{\frac{1+\lambda_1}{Da}}\right) - \sqrt{\frac{1+\lambda_1}{Da}}} \right] \tanh\left(\sqrt{\frac{1+\lambda_1}{Da}}\right)$$

The normalized shear stress at the wall $\bar{\tau}_w$, is given by

$$(31) \quad \bar{\tau}_w = \frac{\tau'_w}{(\tau'_w)_n}$$

$$(32) \quad \bar{\tau}_w = \frac{\left(\tanh \left(\sqrt{\frac{1+\lambda_1}{Da}} \right) - \sqrt{\frac{1+\lambda_1}{Da}} \right) \tanh \sqrt{\frac{1+\lambda_1}{Da}} H}{\left(\tanh \left(\sqrt{\frac{1+\lambda_1}{Da}} H \right) - \sqrt{\frac{1+\lambda_1}{Da}} H \right) \tanh \sqrt{\frac{1+\lambda_1}{Da}}}$$

4. NUMERICAL RESULTS AND DISCUSSION

Analytical expressions for the resistance to the flow $\bar{\lambda}$ and the wall shear stress $\bar{\tau}_w$ are given by (25) and (32) respectively. In order to explicitly observe the effects of various parameters on these flow variables, numerical computation has been done using Mathematica software and the results are graphically depicted in Figs. 2 - 9. Further, it assumed that $d_1 = 0.2$; $L_1=0.2$; $L_2=0.2$; $d_2=0.6$

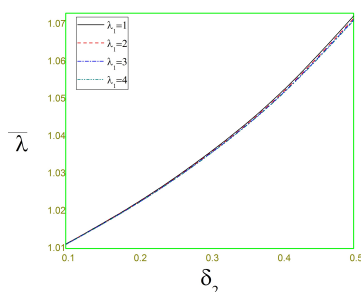


FIGURE 2. Effect of Jeffrey parameter λ_1 on resistance $\bar{\lambda}$ ($\delta_1=0.1$, $Da=0.002$)

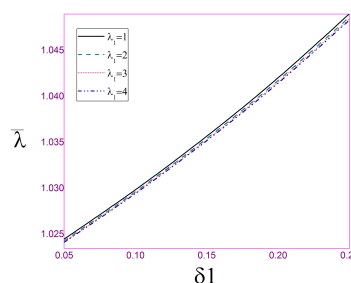


FIGURE 3. Effect of Jeffrey parameter λ_1 on resistance $\bar{\lambda}$ ($\delta_2=0.2$, $Da=0.002$)

Figs. 2 - 5 indicate the effects of various parameters on the resistance to the flow $\bar{\lambda}$. It can be observed that resistance decreases with Jeffrey parameter λ_1 (Figs. 2 and 3) but increases with the permeability (Figs. 4 and 5). This clearly demonstrates an inverse relationship as resistance to the flow sharply decays with an increase in the former whereas it ascends monotonically with a rise in the latter. Further, It can be noticed that the resistance to the flow $\bar{\lambda}$ increases with heights of the Stenoses (δ_1 and δ_2) (Figs. 2 - 5). This result agrees with the previous results obtained by Devanathan and Parvathamma [13] and Joshi et al. [14]. The response to a change in heights of the stenoses exerts a non trivial effect on resistance to the flow and has implications for the effectiveness of the stenotic region.

The effect of different parameters on wall shear stress $\bar{\tau}_w$ is depicted in Figs. 6 -9. The wall shear stress increases with the heights of the stenoses (δ_1 and δ_2) (Figs. 6 -9). This result is expected since increase in height of stenosis leads to decrease in the lumen of the channel so that the stress at the wall increases. Further, the wall shear stress decreases with Jeffrey parameter λ_1 (Figs. 6 and 7) but increases with permeability (Figs. 8 -9). Distinct from earlier graphs in the case of resistance, a significant variation of shear stress is observed with the increase in Jeffrey parameter and permeability.

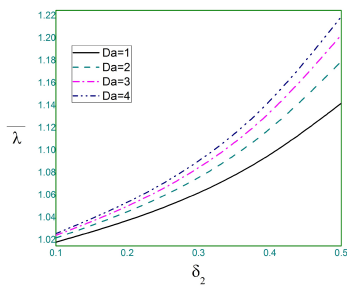


FIGURE 4. Effect of Darcy number on resistance $\bar{\lambda}$ ($\delta_1=0.1$, $\lambda_1=5$)

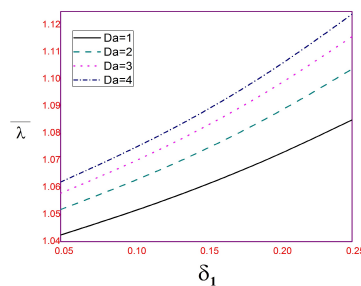


FIGURE 5. Effect of Darcy number on resistance $\bar{\lambda}$ ($\delta_2=0.2$, $\lambda_1=5$)

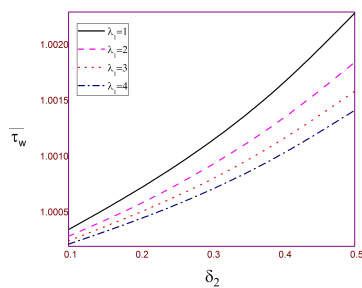


FIGURE 6. Effect of Jeffrey parameter λ_1 on shear stress $\bar{\tau}_w$ ($\delta_1=0.1$, $Da=0.002$)

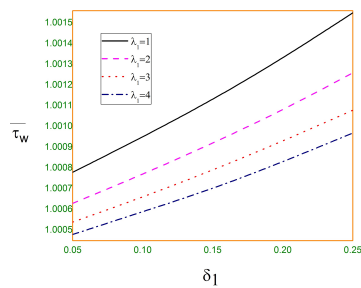


FIGURE 7. Effect of Jeffrey parameter λ_1 on shear stress $\bar{\tau}_w$ ($\delta_2=0.2$, $Da=0.002$)

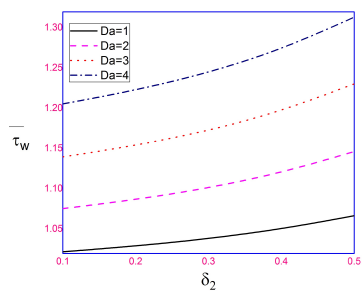


FIGURE 8. Effect of Darcy number Da on shear stress $\bar{\tau}_w$ ($\delta_1=0.1$, $\lambda_1=5$)

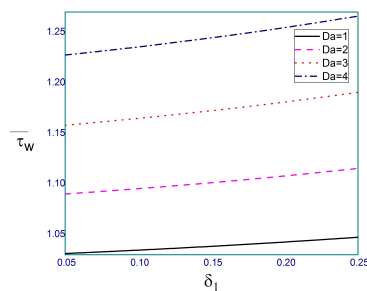


FIGURE 9. Effect of Darcy number Da on shear stress $\bar{\tau}_w$ ($\delta_2=0.2$, $\lambda_1=5$)

5. CONCLUSION

Steady flow of an incompressible Jeffrey fluid through a porous medium with multiple stenoses has been presented. Assuming the stenosis to be mild, closed-form expressions for the resistance to flow and the wall shear stress have

been derived. The effects of various pertinent parameters on these flow characteristics are discussed. It is observed that both the resistance to flow and wall shear stress increases with height of the stenosis but decreases with Jeffrey parameter and Darcy number. The effects of parameters related to heights of stenosis on shear stress can help in better understanding the fluid mechanical aspects in stenotic flow region, which in turn would help in further understanding of the progress and development of arteriosclerosis in physiological systems.

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