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COMPLEMENTS OF A SET AND DE MORGAN'S LAWS VIA IDEALS

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Abstract

In this paper, we introduced L-I-Complements and S-I-complements of a set and study the relationship between these complements with the actual complement of a set. Also we approach De Morgan's laws via ideals.

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1 Introduction

An ideal is a nonempty collection I of subsets of a set satisfying (i) $A \in I$ and $B \subseteq A \Rightarrow B \in I$ (heredity) and (ii) $A \in I$ and $B \in I \Rightarrow A \cup B \in I$ (finite additivity). The subject of ideals in topological spaces has been studied by Kuratowski [3] and Vaidyanathaswamy [4] . Jankovic and Hamlett [2] obtained new topologies using old

ones via ideals and introduced the notation of ideal topological spaces. Immanuel and Manoharan [1] used the concept of ideals to define I-complements of a set and approached De Morgan's laws via ideals.

Definition 1. [1] Let X be a non empty set and I be an ideal on it. For every subset A of X, a set $B \subseteq X$ is said to be an I-complement of A if (i) $A \cup B = X$ and (ii) $A \cap B \in I$.

An I-complement of a set A is denoted by A_I^c .

Result 2. *[1] For any Set A , A^c is an I-complement of the set A.*

Remark 3. *[1] If $I = \{\phi\}$, I-complements of any set coincides with its complement. If I is the power set of a non empty set X, for any subset A of X, the set of all subsets B of X such that $A \cup B = X$ are I-complements.*

2 L-I-Complements and S-I-Complements of a set

De Morgan's laws plays vital role in set theory, Boolean algebra and Propositional logic. In this section, we introduce L-I-complements and S-I- complements of a set and study the relationship between them. Also we approach De Morgan's laws via ideals.

Definition 4. Let X be a nonempty set and I be an ideal on it. For every subset A of X, a set $B \subseteq X$ is said to be an L-I-complement of A if (i) $A^c \cap B^c \in I$ and (ii) $A \cap B = \phi$

We will denote L-I-complements of a set A by A_L^C .

Remark 5. *It is clear that for any Set $A \subseteq X$, A^C is an L-I-complement which gives the assurance of existence of an L-I-complement for the set A.*

Example 6 shows that I-complements and L-I-complements are independent.

Example 6. *Let $X = \{a, b, c\}$ and $I = \{\phi, \{a\}\}$. For the set $A = \{a, b\}$, $\{a, c\}$is an I-Complement but not an L-I-complement.*

Let $X = \{a, b, c\}$ and $I = \{\phi, \{a\}\}$. For the set $A = \{c\}$, $\{b\}$is L-I-Complement but not I-complement.

Remark 7. *For any set $A \subseteq X$, A^c is an L-I-complement but not the converse. Let $X = \{a, b, c\}$ and $I = \{\phi, \{a\}\}$. For the set $A = \{b\}$, $\{c\}$ and $\{a, c\}$ are L-I-complements but the set $\{c\}$is not complement of A.*

The following example shows that *L-I*-complement of a set is not unique.

Example 8. *Let $X = \{a, b, c\}$ and $I = \{\phi, \{a\}\}$. For the set $A = \{b\}$, $\{c\}$ and $\{a, c\}$ are L-I-complements.*

Remark 9. *If $I = \{\phi\}$, L-I-complements of any set coincides with its complement.*

Proposition 10. *Let X be a nonempty set with an ideal I on it. For any set A, if B_1 and B_2 are L-I-complements, then $B_1 \cup B_2$ is also an L-I-complement of A.*

Proof. Since B_1 and B_2 are *L-I*-complement, $A^c \cap B_1^c \in I$ and $A^c \cap B_2^c \in I$. Also $A \cap B_1 = \phi$ and $A \cap B_2 = \phi$. Now

$$\begin{aligned} A^c \cap (B_1 \cup B_2)^c &= A^c \cap (B_1^c \cap B_2^c) \\ &= (A^c \cap B_1^c) \cap B_2^c \\ &= I \cap B_2^c \in I \\ A \cap (B_1 \cup B_2) &= (A \cap B_1) \cup (A \cap B_2) = \phi \cup \phi = \phi \end{aligned}$$

Therefore $B_1 \cup B_2$ is also *L-I*-complement of A. □

Proposition 11. *Let X be a nonempty set with an ideal I on it. For any set A, if B_1 and B_2 are L-I-complements of A, then $(B_1 \cap B_2)$ is also an L-I-complement of A.*

Proof. Since B_1 and B_2 are *L-I*-complement, $A^c \cap B_1^c \in I$ and $A^c \cap B_2^c \in I$.

Also, $A \cap B_1 = \phi$ and $A \cap B_2 = \phi$.

Now

$$\begin{aligned} A^c \cap (B_1 \cap B_2)^c &= A^c \cap (B_1^c \cup B_2^c) \\ &= (A^c \cap B_1^c) \cup (A^c \cap B_2^c) \in I \\ A \cap (B_1 \cap B_2) &= (A \cap B_1) \cap B_2 \\ &= \phi \cap B_2 \\ &= \phi. \end{aligned}$$

Therefore $B_1 \cap B_2$ is L-I-complement of A. □

Corollary 12. *Let X be a non empty set with an ideal I on it. For any set A, if $A_1, A_2, \ldots, A_n$ are L-I-complements, then*
(i) $(A_1 \cup A_2 \cup \cdots \cup A_n)$ is an L-I-complement of A.
(ii) $(A_1 \cap A_2 \cdots \cap A_n)$ is an L-I-complement of A.

Proposition 13. *Let I_1 and I_2 be two ideals on a non empty set X with $I_1 \subseteq I_2$. Then for any set $A \subseteq X$,every L-I_1-complement of A is also an L-I_2-complement .*

Definition 14. Let X be a nonempty set and I be an ideal on it. For every subset A of X, a set $B \subseteq X$ is said to be an S-I-complement of A if (i) $A^c \cap B^c \in I$ and (ii) $A \cap B \in I$.

We will denote S-I-complement of a set A by A_S^C.

Example 15. *S-I-complement of a set is not unique. Let $X = \{a, b, c\}$ and $I = \{\phi, \{a\}\}$. For the set $A = \{c\}$, the sets $\{b\}$ and $\{a, b\}$ are S-I-complements.*

Remark 16. *For any set $A \subseteq X$, A^c is an S-I-complement but not the converse. Let $X = \{a, b, c\}$ and $I = \{\phi, \{a\}\}$. For the set $A = \{a, b\}$, the set $\{a, c\}$ is an S-I-complement which is not complement of A.*

Proposition 17. *Let X be a nonempty set and I be an ideal on it. Let $A \subseteq X$. A set $B \subseteq X$ is an L-I-complement of A then it is an S-I-complement of A.*

Remark 18. *The following example shows that the Converse of the above Proposition need not be true.*

Example 19. *Let $X = \{a, b, c\}$ and $I = \{\phi, \{a\}\}$. For*

the set $A = \{a, c\}$, $\{a, b\}$ is an S-I-complement but not an L-I-complement.

Proposition 20. *Let X be a nonempty set with an ideal I on it. For any set A, if B_1 and B_2 are S-I-complements, then $B_1 \cup B_2$ is also an S-I-complement of A.*

Proposition 21. *Let X be a nonempty set with an ideal I on it. For any set A, if B_1 and B_2 are S-I-complements of A, then $(B_1 \cap B_2)$ is also an S-I-complement of A.*

Corollary 22. *Let X be a non empty set with an ideal I on it. For any set A, if $A_1, A_2, \ldots, A_n$ are S-I-complements , then*
(i) $(A_1 \cup A_2 \cup \cdots \cup A_n)$ is an S-I-complement of A;
(ii) $(A_1 \cap A_2 \cdots \cap A_n)$ is an S-I-complement of A.

Proposition 23. *Let I_1 and I_2 be two ideals on a non empty set X with $I_1 \subseteq I_2$. Then for any set $A \subseteq X$,every S-I_1-complement of A is also an S-I_2-complement .*

Theorem 24. *Let X be a nonempty set and I be an ideal on it. For every subset A of X,*
(i) one of the L-I-complements of A_L^C is A;
(ii) one of the S-I-complements of A_S^C is A.

Proof. Follows from the definitions of L-I-complements and S-I-complements of A. □

Theorem 25. *Let X be a nonempty set and I be an ideal on it.If $I = \{\phi\}$,then the following are equivalent*
Let $A \subseteq X$, then
(i) B is an complement of A;
(ii) B is an L-I-complement of A;
(iii) B is an S-I-complement of A;
(iv) B is an I-complement of A.

Proposition 26. *Let $A \subseteq X$. Let B is both L-I-complement and I-complement of A. Then B is an S-I-complement of A.*

The following example shows that the converse of the above proposition is not true.

Example 27. *Let $X = \{a, b, c\}$ and $I = \{\phi, \{a\}\}$. For the set $A = \{a, c\}$, the set $\{a, b\}$ is an S-I-complement which is an I-complement but not an L-I-complement. For the set $A = \{b\}$, the set $\{c\}$is an S-I-complement which is also an L-I-complement but not an I-complement.*

From the observations, we have the following diagram in which none of the implications is reversible.

$$\begin{array}{ccc} Complement & \longrightarrow & L{-}Complement \\ \downarrow & & \downarrow \\ I{-}Complement & \longrightarrow & S{-}Complement \end{array}$$

Theorem 28. *Let X be a non-empty set with ideal I on it. Let A and B be two sub sets of X. then*
(i) An L-I-complement of $(A \cup B)$ is $(A_L^C \cap B_L^C)$
(ii) An L-I-complement of $(A \cap B)$ is $(A_L^C \cup B_L^C)$

Proof. (i)

$$\begin{aligned} (A \cup B)^C \cap (A_L^C \cap B_L^C)^C &= X - ((A \cup B) \cup ((A_L^C \cap B_L^C)) \\ &= X - ((A \cup B \cup A_L^C) \cap (A \cup B \cup B_L^C)) \\ &= (A^C \cap B^C \cap (A_L^C)^C) \cup (A^C \cap B^C \cap (B_L^C)^C) \\ &\in I. \\ (A \cup B) \cap (A_L^C \cap B_L^C) &= ((A \cup B) \cap A_L^C) \cap B_L^C \\ &= ((A \cap A_L^C) \cup (A_L^C \cap B)) \cap B_L^C \\ &= (A \cap A_L^C \cap B_L^C) \cup (A_L^C \cap B \cap B_L^C) \\ &= \phi \cup \phi = \phi. \end{aligned}$$

ii) Proof is similar to (i). □

Corollary 29. ((De Morgan's laws) *Let X be a non-empty set with ideal I on it. Let A and B be two sub sets of X. then*
(i) $(A \cup B)^C = A^C \cap B^C$
(ii) $(A \cap B)^C = A^C \cup B^C$

Proof. Choosing $I = \{\phi\}$ in the above theorem, we can prove the results. □

Theorem 30. *Let X be a non-empty set with ideal I on it. Let A and B be two sub sets of X. then*

(i) An S-I-complement of $(A \cup B)$ *is* $(A_S^C \cap B_S^C)$
(ii) An S-I-complement of $(A \cap B)$ *is* $(A_S^C \cup B_S^C)$

Proof. Similar to the proof of theorem 28 □

Corollary 31. ((De Morgan's laws) *Let* X *be a non-empty set with ideal* I *on it. Let* A *and* B *be two sub sets of* X*. then*
(iii) $(A \cup B)^C = A^C \cap B^C$
(iv) $(A \cap B)^C = A^C \cup B^C$

Proof. Choosing $I = \{\phi\}$ in the above theorem, we can prove the results. □

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