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On γ^{α}-regular spaces, γ^{α}-normal spaces in topological spaces

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Abstract

In this paper the concept of γ^{α}-regular spaces and γ^{α}-normal spaces have been introduced and the relationship among them are studied.

Key words: γ^{α}-regular space, γ^{α}-normal space, α-γ-Locally closed sets
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1 Introduction

O.Najastad [9] introduced α- open sets in a topological space and studied some of its properties. Kasahara [8] defined the concept of an operation on topological spaces and introduced α - closed graphs of an operation. Ogata [10] called the operation α as γ operation and introduced the notion of τ_{γ} which is the collection of all γ- open sets in a topological space (X,τ). N.Kalaivani and G.Sai sundara krishnan [7] introduced α-γ-open sets by using the concept of γ-open sets in a topological space. They studied (α-γ, β)- continuous mappings and α-(γ, β)-continuous mappings. They extended their studies on α-(γ, β)-open(closed) mappings and γ-generalized α-open sets.

In this paper in section 3 we introduce the γ^{α}-regular space and study some of their properties.

In section 4 we introduce the concept γ^{α}-normal space and analyze some of their properties.

2 Preliminaries

In this section we recall some of the basic Definitions .

Definition 2.1 [9] Let(X,τ) be a topological space and A be a subset of X. Then A is said to be α-open set if A $\subseteq$ int(cl(int(A))).

Definition 2.2 [10] Let(X,τ) be a topological space, an operation γ on τ is a mapping from τ on to the power set P(X) of X such that $V \subseteq V^{\gamma}$ for each V $\in \tau$,where V^{γ} denotes the value of γ at V.

Definition 2.3 [10] Let (X,τ) be a topological space and A be a subset of X and γ be an operation on τ. Then A is said to be a γ- open set if for each x $\in$ A there exists an open set U such

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that x ∈ U and $U^\gamma \subseteq A$. τ_γ denotes the set of all γ- open sets in (X,τ).

Definition 2.4 (i) [11] Let (X,τ) be a topological space and γ be an operation on τ .Then τ_γ - interior of A is defined as union of all γ- open sets contained in A and it is denoted $\tau_\gamma - int(A)$. That is $\tau_\gamma - int(A) = \bigcup \{U : U \text{is a} \gamma - \text{open set and} U \subseteq A\}$

(ii) [10] Let (X,τ) be a topological space and γ be an operation on τ. Then τ_γ -closure of A is defined as intersection of all γ- closed sets containing A and it is denoted by $\tau_\gamma - cl(A)$. That is $\tau_\gamma - cl(A) = \bigcap \{F : F \text{is a} \gamma - \text{closed set and} A \subseteq F\}$

Definition 2.5 [7] Let (X,τ) be a topological space and γ be an operation on τ. Then a subset A of X is said to be a α-γ-open set if and only if $A \subseteq \tau_\gamma - int(\tau_\gamma - cl(\tau_\gamma - int(A)))$

Definition 2.6 [7] Let (X,τ) be a topological space and γ be an operation on τ. Then a subset A of X is said to be a α-γ-closed if and only if X − A is α-γ-open.

Definition 2.7 [7] Let(X ,τ) be a topological space and γ be an operation on τ and A be a subset of X . Then $\tau_{\alpha-\gamma}$- interior of A is the union of all α-γ-open sets contained in A and it is denoted by $\tau_{\alpha-\gamma} - int(A)$. $\tau_{\alpha-\gamma}$-int (A) $= \cup \{U : U \textit{ is a } \alpha - \gamma - \textit{open set and } U \subseteq A\}$

Definition 2.8 [7] Let (X ,τ) be a topological space and γ be an operation on τ. Let A be a subset of X . Then $\tau_{\alpha-\gamma}$-closure of A is the intersection of α-γ- closed sets containing A and it is denoted by $\tau_{\alpha-\gamma}$- cl(A). That is $\tau_{\alpha-\gamma}$- cl(A) $= \cap \{F : F \textit{ is a } \alpha - \gamma - \textit{closed set and } A \subseteq F\}$

3 γ^α-regular spaces

Definition 3.1 A topological space X is said to be a γ^α-regular space, if for any α-γ-closed set A and x ∉ A, there exist γ-open sets U ,V such that x ∈ U, A ⊆ V and $U \cap V = \phi$.

Example 3.1 Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Define an operation γ on τ such that A^γ = cl(A) if b ∉ A for every A ∈ τ and A^γ = A if b ∈ A for every A ∈ τ. Then$\tau_\gamma = \{\phi, X, \{b\}, \{a, b\}, \{a, c\}\}$ and $\tau_{\alpha-\gamma} = \{\phi, X, \{b\}, \{a, b\}, \{a, c\}\}$.
Here X is a γ^α-regular space but X is not a regular space.

Theorem 3.1 Every γ^α-regular space X is an α-γ-regular space.But the converse need not be true.
ProofProof follows from the Definition 3.1 and Theorem 2.1.[10]

Theorem 3.2 The following statements about a topological space (X,τ) are equivalent:
(*i*) X is a γ^α-regular space. (*ii*) For any α-γ-open set U in X, there is a γ-open set V containing x such that x ∈ $\tau_\gamma - cl(V) \subseteq U$.

Proof (*i*) ⇒ (*ii*) Let U be an α-γ-open set and x ∈ U. Then X − U is an α-γ-closed set such that x ∉ $X - U$. By the regularity of X,there exist γ-open sets V_1, V_2 such that x ∈ V_1, $X - U \subseteq V_2$ and $V_1 \cap V_2 = \phi$.Then $X - V_2$ is a γ-closed set contained in U. Then $V_1 \subseteq X - V_2 \subseteq U$. This implies that $\tau_\gamma - cl(V_1) \subseteq X - V_2 \subseteq U$. Consequently, x ∈ V and $\tau_\gamma - cl(V) \subseteq U$.

$(ii) \Rightarrow (i)$ Let A be an α-γ-closed set which does not contain x. By (ii) there exists an α-γ-open sets U and V such that x $\in$ V, x $\in \tau_\gamma - cl(V) \subseteq U$. Thus x$\in$ V, A$\subseteq$ U and U $\bigcap$ V= ϕ. Hence X is a γ^α-regular space.

Theorem 3.2 A topological space X is a γ^α-regular space if for each x $\in$ X and an α-γ- closed set A such that x $\notin$ A , there exists γ-open sets U,V in X such that x $\in$ U and A $\subseteq$ V and $\tau_\gamma - cl(U) \cap \tau_\gamma - cl(V) = \phi$.

Proof Let x $\in$ X and A be an α-γ-closed set such that x $\notin$ A , by Theorem 3.1 there is a γ-open set W such that x $\in$ W , $\tau_\gamma - cl(W) \subseteq X - A$. By Theorem 3.1 ,there is an γ-open set U containing x such that $\tau_\gamma - cl(U) \subseteq W$. Let $V = X - \tau_\gamma - cl(W)$. Then $\tau_\gamma - cl(U) \subseteq W \subseteq \tau_\gamma - cl(W) \subseteq X - A$ implies that $A \subseteq X - \tau_\gamma - cl(W) = V$. Also $\tau_\gamma - cl(U) \cap \tau_\gamma - cl(V) = \tau_\gamma - cl(U) \cap \tau_\gamma - cl(X - \tau_\gamma - cl(W)) \subseteq W \cap \tau_\gamma - cl(X - \tau_\gamma - cl(W)) \subseteq \tau_\gamma - cl(W \cap (X - \tau_\gamma - cl(W))) = \tau_\gamma - cl(\phi) = \phi$.

Theorem 3.3 Let X be a topological space and γ be an operation on τ. Then the following properties are equivalent:
(i) X is a γ^α-regular space;
(ii) For each α-γ-closed set F and each point x $\in X - F$ there exists an U $\in \tau_\gamma$ and a γ-open set V such that x $\in$ U, F $\subseteq$ V and U $\bigcap V = \phi$;
(iii) For each α-γ-closed set F of X, F = $\bigcap \{\tau_\gamma - cl(V) \subseteq V and V is an \gamma$ g -open set;
(iv) For each subset A of X and each α-γ-closed set F such that A $\bigcap F = \phi$, there exists an U $\in \tau_\gamma$ and a γg-open set V such that A $U \neq \phi$, F $\subseteq$ V and U $\bigcap V = \phi$.
Proof $(i) \Rightarrow (ii)$ Proof follows from the Definition 3.1. and by Theorem 2.4.
$(ii) \Rightarrow (iii)$ For any α-γ-closed set F, F $\subseteq \bigcap \{\tau_\gamma - cl(V) \subseteq V and V is an \gamma$ g -open set. Suppose that x $\in$ X - F. By the assumption, there exists a γ-open set U and a γg-open set V such that x $\in$ U, F $\subseteq$ V and U $\bigcap V = \phi$. By Theorem 2.1,2.5, x $\in X - \tau_\gamma - cl(V)$ and hence x $\in X - \{\bigcap \tau_{\alpha-\gamma} - cl(V) and V is an \alpha - \gamma g - open set\}$.Therefore, F $\supseteq \bigcap \{\tau_{\alpha-\gamma} - cl(V) : F \subseteq V and V is an \alpha - \gamma g - open set\}$ and hence F =$\bigcap \{\tau_{\alpha-\gamma} - cl(V) : F \subseteq V is an \alpha - \gamma g - open set\}$.
$(iii) \Rightarrow (iv)$ Let F be any α-γ-closed set of X and A be a nonempty subset of X such that A $\bigcap F = \phi$. Take a point x $\in X - \tau_\gamma - cl(V)$. Let U = X -$\tau_\gamma - cl(V)$, then U $\in \tau_\gamma$, A $\bigcap U \neq \phi$ and U $\bigcap V = \phi$.
$(iv) \Rightarrow (i)$ Let F be any α-γ-closed set of X and x $\in X - F$. Then $\{x\} \bigcap F = \phi$ and there exists U $\in \tau_\gamma$ and a γg-open set W such that x $\in$ U, F $\subseteq$ W and U $\bigcap W = \phi$. Put V = $\tau_\gamma - int(W)$, then by Definition 3.1, F $\subseteq$ V, V $\in \tau_\gamma$ and U $\bigcap V = \phi$. Therefore, X is a γ^α-regular space.

Theorem 3.4 If $f : (X, \tau) \to (Y, \sigma)$ is (γ, β)-irresolute,(γ, β)-open mapping, (γ, β)-g-closed surjection and X is a γ^α-regular space, then Y is a β^α-regular space.
Proof Let y $\in$ Y and V be a β-open set of Y containing y. Let x be a point of X such that y= f(x). Since X is an γ^α-regular space, there exists a γ-open set U of X such that x $\in U \subseteq \tau_\gamma - cl(U) \subseteq f^{-1}(V)$. This implies that y $\in f(U) \subseteq f(\tau_\gamma - cl(U)) \subseteq$ V. Since f is (γ, β)-open and (γ, β)-g-closed mapping, f(U) $\in \sigma_\beta$ and $f(\tau_\gamma$-cl(U)) is a β-g-closed set in Y. Then Y $- f(\tau_\gamma$-cl(U)) is a β-g-open set containing Y $-$ V . Therefore, $\bigcap \{\sigma_\beta - cl(Y - f(\tau_\gamma - cl(U)) : Y - V \subseteq Y - f(\tau_\gamma - cl(U)) and Y - f(\tau_\gamma - cl(U)) is a \beta$ = Y $-$ V. Hence by Theorem 3.3, Y is an β^α-regular space.

4 γ^{α}-normal space

Definition 4.1 A space X is said to be a γ^{α}-normal space, if for any disjoint α-γ-closed sets A ,B of X, there exist γ-open sets U ,V such that A $\subseteq$ U, B $\subseteq$ V and $U \cap V = \phi$.

Example 4.1 Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$. Define an operation γ on τ such that A^{γ}=cl(A) if b $\notin$ A for every A $\in \tau$ and A^{γ}=A if b *in* A for every A $\in \tau$. Then$\tau_{\gamma} = \{\phi, X, \{c\}, \{a, b\}, \{a, c\}\}$ and $\tau_{\alpha-\gamma} = \{\phi, X, \{c\}, \{a, b\}, \{a, c\}\}$.Here X is a γ^{α}-normal space.

Remark 4.1 Every γ-normal space X is a γ^{α}-normal space.But the converse need not be true.

Proof Proof follows from the Definition 4.1 and Theorem 2.1.

Theorem 4.1 For a topological space X the following statements are equivalent: (i) X is a γ^{α}-normal space. (ii) For every pair of γ-open sets U and V whose union is X , there exist γ *g*-closed sets A and B such that A $\subseteq$ U and B$\subseteq$ V and A $\bigcup$ B = X. (iii) For every α-γ-closed set H and every γ-open set K containing H, there exists a γ *g*-open set U such that H $\subseteq$ U $\subseteq$ τ_{γ}-cl(U)$\subseteq$ K.
Proof $(i) \Rightarrow (ii)$ Let U and V be a pair of γ-open sets in a γ^{α}-normal space X such that X = U $\cup$ V. Then X - U , X - V are disjoint α-γ-closed sets. Since X is a γ^{α}-normal space, there exists disjoint γ-open sets U_1 and V_1 such that X - U $\subseteq U_1$ and X - V $\subseteq V_1$. Let A = X - U_1, B = X - V_1. Then A and B are γ *g*-closed sets such that A$\subseteq$ U, B $\subseteq$ V and A$\cup$ B = X. $(ii) \Rightarrow (iii)$ Let H be an α-γ-closed set and K be a γ-open set containing H. Then X - H and K are γ-open sets whose union is X. Then by (ii) there exists γ-closed sets M_1 and M_2 such that $M_1 \subseteq X - H$ and $M_2 \subseteq K$ and $M_1 \cup M_2$ = X. Then H$\subseteq$ X-M_1 , K$\subseteq$ X - M_2 and (X-M_1) $\cap$ $(X - M_2) = \phi$. Let U = X- M_1 and V= X- M_2. Then U and V are disjoint γ-open sets such that x $\in$ X. Then for every γ-open set V containing y such that y $\in$ V ,$\tau_{\gamma} - cl(V) \subseteq$ U. Thus for each y $\in$ X, the set τ_{γ}-cl(V) is a γ-neighbourhood of X. $(iii) \Rightarrow (i)$ Proof follows from the 4.1.Definition.
Theorem 4.2 A topological space X is a γ^{α}-normal space if and only if for any α-γ-closed set A and γ-open set U containing A, there exists a γ-open set V containing A such that A $\subseteq V \subseteq \tau_{\gamma} - cl(V) \subseteq$ U.
Proof Since U is an α-γ-open set containing A, X-U is an α-γ-closed set and A$\cap(X - U) = \phi$. Since X is a γ^{α}-normal spcae, there exists γ-open sets V and V_1 such that A$\subseteq$ V, X-U$\subseteq V_1$ and V$\cap V_1 = \phi$. Hence A$\subseteq V \subseteq \tau_{\gamma} - cl(V) \subseteq \tau_{\gamma} - cl(X - V_1) = X - V_1 \subseteq$ U. Conversely, let A, B be disjoint α-γ-closed sets in X. Then A $\subseteq$ X -B, where X -B is an α-γ-open set in X. By asumption there is a γ-open set V such that $\tau_{\gamma} - cl(V) \subseteq$ X-B, which implies B $\subseteq X - \tau_{\gamma} - cl(V)$ and V$\cap(X - \tau_{\gamma} - cl(V)) = \phi$. Hence A$\subseteq$ V, B$\subseteq X - \tau_{\gamma} - cl(V)$. This proves that X is a γ^{α}-normal space.
Theorem 4.3 A γ^{α}-normal and α-γ-T_1 space is a γ^{α}-regular space.
Proof Suppose A is an α-γ-closed set and x$\notin$ A. Since X is an α-γ-T_1 space, by 2.5.Theorem [6], each $\{x\}$is an α-γ-closed set in X. Since X is a γ^{α}-normal space, there exist γ-open sets U, V such that $\{x\} \subseteq$ U, A $\subseteq$ V, U$\cap V = \phi$ or x$\in$ U, A$\subseteq$ V and U$\cap V = \phi$ implies that X is a γ^{α}-regular space.
Theorem 4.4 Let $f : X \to Y$ be a (γ,β)-closed , α-(γ,β)-continuous mapping from X onto Y. If X is a γ^{α}-normal space, then Y is a β^{α}-normal space, where β is an open operation.
Proof Let A_1,B_1 be disjoint $\alpha - \beta$-closed subsets of Y. Then by α-(γ,β)-continuity of f, A = $f^{-1}(A_1)$, B =$f^{-1}(B_1)$, are disjoint α-γ-closed subsets of X. Since X is a γ^{α}-normal space, there exist γ-open sets U and V such that A$\subseteq$ U, B$\subseteq$ V and U$\cap V = \phi$. Since f is (γ,β)-closed mapping,

f(x-U) and f(X-V) are β-closed sets in Y. Then $U_1 = Y - f(X - U)$, $U_2 = Y - f(x - V)$ are disjoint β-open sets in Y containing A_1 ,B_1 respectively. This proves that Y is a β^α-normal space.

Theorem 4.5 A topological space X is a γ^α-normal spce if and only if for each pair A , B of disjoint α-γ-closed sets in X, thete exists γ-open sets U, V in X such that A $\subseteq$ U , B $\subseteq$ V and $\tau_\gamma - cl(U) \cap \tau_\gamma - cl(V) = \phi$.

Proof Let A be an α-γ-closed set and B be an α-γ-closed set not containing A. Then X - B is an α-γ-open set and A$\subseteq$ X - B. Then by 4.2 Theorem, there exists a γ-open set C such that A$\subseteq$ C $\subseteq$ τ_γ-cl(C) $\subseteq$ X-B. Since τ_γ-cl(C)$\subseteq$ X - B, again by 4.2. Theorem, there exits a γ-open set U containing A such that τ_γ-cl(C)$\subseteq$ C. Consequently, A $\subseteq U \subseteq \tau_\gamma - cl(U) \subseteq C \subseteq \tau_\gamma - cl(C) \subseteq$ X - B. Then $\tau_\gamma - cl(C) \subseteq$ X - B implies that B $\subseteq X\tau_\gamma$-cl(C). Let V = X - τ_γ-cl(C). Then V is a γ-open set containing B and $\tau_\gamma - cl(U) \cap \tau_\gamma - cl(V) = \tau_\gamma - cl(U) \cap (X - \tau_\gamma - cl(C)) \subseteq C \cap \tau_\gamma - cl(X - \tau_\gamma - cl(C)) \subseteq \tau_\gamma - cl(C \cap X - \tau_\gamma - cl(C)) = \tau_\gamma - cl(\phi) = \phi$. Thus U, V are the required -open sets in X. The converse part follows fron the 2.13, 4.1. Definitions.

Definition 4.6 A topological space X is said to be an ultra γ^α-normal space , if for any distinct α-γ-closed sets A and B of X, there exists γ-clopen sets, U, V such that A $\subseteq$ U and B $\subseteq$ V and U $\cap$ V = ϕ.

Theorem 4.6 If $f : X \to Y$ is a contra α-γ-continuous ,closed injective mapping and Y is an ultra γ^α-normal spacd, then X is an γ^α-normal space.

Proof Let F_1 and F_2 be disjoint closed subsets of X. Since f is a closed and injective mapping, $f(F_1)$ and $f(F_2)$ are disjoint closed subsets of Y.Since Y is an ultra γ^α-normal space,$f(F_1)$ and $f(F_2)$ are separated by disjoint γ-clopen sets V_1 and V_2 respectively. Hence $F_i \subseteq f^{-1}(V_i)$, $f^{-1}(V_i) \in \tau_{\alpha-\gamma}$, for i=1,2. and $f^{-1}(V_1) \cap f^{-1}(V_2) = \phi$. Thus X is a γ^α-normal space.

Theorem 4.7 Let $f : X \to Y$ be a (γ, β)-irresolute ,(γ, β)-g-closed surjection and X is a γ^α-normal space, then Y is a β^α-normal space.

Proof Let A_1, B_1 be disjoint β-closed subsets of Y. Then A =$f^{-1}(A_1)$, B =$f^{-1}(B_1)$, are disjonit γ-closed subsets of X, since f is a (γ, β)-irresolute mapping. Since X is a γ^α-normal space, there exist γ-open sets U and V such that A $\subseteq$ U, B$\subseteq$ V and U$\cap V = \phi$. Since f is a (γ, β)-g- closed mapping, f(X-U) and f(X-V) are β-closed sets in Y. Then$U_1 = Y - f(X - U)$, $U_2 = Y - f(X - V)$ are disjoint β-open sets in Y containing A_1, B_1 respectively. This proves that Y is a β^α-normal space.

Theorem 4.8 Let $f : X \to Y$ be a (γ, β)-continuou mapping , (γ, β)-g-closed surjection and X is a γ^α-normal space, then Y is β^α-normal space.

Proof Let A_1, B_1 be disjoint β-closed subsets of Y.Since f is a (γ, β)-continuou mapping, A = $f^{-1}(A_1)$,B =$f^{-1}(B_1)$, are disjoint γ-closed subsets of X. Since X is a γ^α-normal spce, there exist γ-open sets U and V such that A$\subseteq$ U , B$\subseteq$ V and U$\cap V = \phi$. Since f is a (γ, β)-closed mapping, f(X-U) and f(X-V) are β-closed sets also,α-β-closed sets in Y. Then $U_1 = Y - f(X - U)$,$U_2 = Y - f(X - V)$) are disjoint β-open sets in Y containing A_1, B_1 respectively.This proves that Y is a β^α-normal space.

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