d-LUCKY LABELING OF ARBITRARY SUPER SUBDIVISION OF SOME GRAPHS

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Abstract

A function \( l : V(G) \to N \) is said to be \( d \)-lucky labeling if there exists a function \( c : V(G) \to N \) such that \( c(u) = d(u) + \sum_{v \in N(u)} l(v) \), where \( d(u) \) denotes the degree of \( u \) and \( N(u) \) denotes the neighborhood of \( u \) and \( c(u) \neq c(v) \) for every pair of adjacent vertices \( u \) and \( v \) in \( G \). The \( d \)-lucky number of a graph is denoted by \( \eta_{dl}(G) \), is the least positive \( k \) such that \( G \) has a \( d \)-lucky labeling from the set \( \{1, 2, \ldots, k\} \). In this paper we obtain \( \eta_{dl}(G) = 2 \) for arbitrary super subdivision of path, cycle, star, wheel and corona of \( P_n \) graphs.

AMS Subject Classification: 05C78.

Key Words and Phrases: \( d \)-Lucky labeling, \( d \)-lucky number, super subdivision.
1 Introduction

Graph coloring is one of the major research areas of study in graph theory. It is an assignment of labeling graph components such as vertices, edges or regions under some constraints. Francis Guthrie first conjectured the Four color theorem which is considered as the first result about Graph coloring and it was first proved by Kenneth Appel and Wolfgang Haken in [1]. Some of the important applications of Graph coloring are pattern matching, sports scheduling, solving sudoku puzzles etc. Mirka Miller et al. introduced the concept of $d$-Lucky labeling [5] and is defined as follows. A function $l : V(G) \to N$ is said to be $d$-lucky labeling if there exists a function $c : V(G) \to N$ such that $c(u) = d(u) + \sum_{v \in N(u)} l(v)$, where $d(u)$ denotes the degree of $u$ and $N(u)$ denotes the neighborhood of $u$ and $c(u) \neq c(v)$ for every pair of adjacent vertices $u$ and $v$ in $G$. A graph which admits $d$-lucky labeling is the $d$-lucky labeled graph. The $d$-lucky number of a graph is denoted by $\eta_{dl}(G)$, is the least positive $k$ such that $G$ has a $d$-lucky labeling from the set $\{1, 2, \ldots, k\}$. The concept of super subdivision of graphs was introduced by Sethuraman and Selvaraju [7]. It is defined as follows : Let $G$ be a graph with $p$ vertices and $q$ edges. A graph $H$ is said to be an arbitrary super subdivision of $G$ if $H$ is obtained from $G$ by replacing each edge $e_i$ by a complete bipartite graph $K_{2,m_i}$ (where $m_i$ is any positive integer and may vary for each edge arbitrarily) in such a way that the ends of each edge $e_i$ are merged with the two vertices of 2-vertices part of $K_{2,m_i}$ after removing the edge from $G$ and it is denoted by $ASS(G)$. In the literature, It is proved that the arbitrary super subdivision of graphs admit graceful labeling [2],[4]. In this work $P_n$ is the path graph of length $n$ and a closed path is called a cycle and a cycle of length $n$ is denoted by $C_n$ and the complete bipartite graph of the form $K_{1,n-1}$ is a star graph with $n$ vertices and it is denoted by $S_n$ also the wheel graph $W_n$ is defined as $K_1 + C_{n-1}$ where $K_1$ is the singleton graph and the corona of path graph $P_n$ is obtained from $P_n$ by attaching a pendent vertex to each vertex of $P_n$ it is denoted by $P_n^\ast$. For all terminologies refer [3] and [6]. In this paper we prove that the arbitrary super subdivision of path, cycle, star, wheel and corona of $P_n$ graphs admit $d$-lucky labeling and obtain their $d$-lucky numbers.
2 Main Results

Theorem 1. The arbitrary super subdivision of path graph \( P_n \) is a \( d \)-lucky labeled graph and the \( d \)-lucky number is \( \eta_{dl}(\text{ASS}(P_n)) = 2 \).

Proof. Let \( P_n \) be a path graph whose vertex set is \( V = \{v_0, v_1, v_2, \ldots, v_n\} \) and the edge set is \( E = \{e_{i+1} = v_i v_{i+1} / 0 \leq i \leq n - 1\} \). Let \( \text{ASS}(P_n) \) be an arbitrary super subdivision of a path graph \( P_n \). This graph has \((m_1 + m_2 + \cdots + m_n) + n + 1\) vertices and \( 2(m_1 + m_2 + \cdots + m_n) \) edges.

Denote the vertex set of \( \text{ASS}(P_n) \) as \( V = \{v_i \cup v_{i(j)}, v_{i(i+1)} \cup v_{n} / 1 \leq j \leq m_{i+1}, 0 \leq i \leq n - 1\} \) and the edge set as \( E = \{v_i v_{i(j)} \cup v_{i(i+1)}v_{i+1} / 1 \leq j \leq m_{i+1} \text{ and } 0 \leq i \leq n - 1\} \).

Define a function \( l : V(\text{ASS}(P_n)) \rightarrow \mathbb{N} \) by \( l(v_i) = 1 \), \( l(v_{i(i+1)}) = 2 \) for \( 1 \leq j \leq m_{i+1}, 0 \leq i \leq n - 1 \) and \( l(v_n) = 1 \).

Now \( c(u) \)'s are calculated as follows:

\[
\begin{align*}
  c(v_0) &= d(v_0) + \sum l(N(v_0)) = m_1 + 2m_1 = 3m_1 \\
  c(v_i) &= \begin{cases} 
    3(m_i + m_{i+1}) & \text{for } 1 \leq i \leq n - 1, \\
    3m_{i+1} & \text{for } 1 \leq j \leq m_{i+1}, \end{cases}
\end{align*}
\]

Thus \( c(u) \neq c(v) \) for any two adjacent vertices in \( \text{ASS}(P_n) \). Hence the arbitrary super subdivision of path graph admits \( d \)-lucky labeling with \( \eta_{dl}(\text{ASS}(P_n)) = 2 \). \( \square \)

Example 2.

![Figure 1: \( d \)-lucky labeling of arbitrary super subdivision of path graph \( \text{ASS}(P_4) \)](image)

Theorem 3. The arbitrary super subdivision of cycle graph \( C_n \) is a \( d \)-lucky labeled graph and the \( d \)-lucky number is \( \eta_{dl}(\text{ASS}(C_n)) = 2 \).
Proof. Let $C_n$ be a cycle graph whose vertex set is $V = \{v_1, v_2, \ldots, v_n\}$ and the edge set is $E = \{e_i = v_iv_{i+1} \cup v_nv_1 \mid 1 \leq i \leq n-1\}$. Let $ASS(C_n)$ be an arbitrary super subdivision of a cycle graph $C_n$. This graph has $(m_1 + m_2 + \cdots + m_n) + n$ vertices and $2(m_1 + m_2 + \cdots + m_n)$ edges.

Denote the vertex set of $ASS(C_n)$ as $V = \{v_i \cup v_{i(i+1)} \cup v_{n1} \mid 1 \leq i \leq n-1, 1 \leq j \leq m_i, 1 \leq k \leq m_n\}$ and the edge set as $E = \{v_iv_{i(i+1)} \cup v_{n1}v_{n1} \mid 1 \leq i \leq n-1, 1 \leq j \leq m_i, 1 \leq k \leq m_n\}$.

Define a function $l : V(ASS(C_n)) \to N$ by $l(v_i) = 1$ & $l(v_{i(i+1)}) = 2$ for $1 \leq i \leq n-1, 1 \leq j \leq m_i$ and $l(v_{n1}) = 1$ & $l(v_{n1}^k) = 2$ for $1 \leq k \leq m_n$. Now $c(u)$’s are calculated as follows:
\begin{align*}
ge(v_i) &= d(v_i) + \sum l(N(v_i)) = 3(m_1 + m_n); \quad c(v_i) = d(v_i) + \sum l(N(v_i)) = 3(m_{i-1} + m_i) \text{ for } 2 \leq i \leq n \text{ and } c(v_{i(i+1)}) = 4 \text{ for } 1 \leq i \leq n-1, 1 \leq j \leq m_i \text{ and } c(v_{n1}^k) = 4 \text{ for } 1 \leq k \leq m_n.
\end{align*}

Thus $c(u) \neq c(v)$ for any two adjacent vertices in $ASS(C_n)$. Hence the arbitrary super subdivision of cycle graph admits $d$-lucky labeling with $\eta_{dl}(ASS(C_n)) = 2$. \hfill \Box

Example 4.

Figure 2: $d$-lucky labeling of arbitrary super subdivision of cycle graph $ASS(C_5)$

Theorem 5. The arbitrary super subdivision of star graph $S_n$ is a $d$-lucky labeled graph and the $d$-lucky number is
\[ \eta_{dl}(ASS(S_n)) = 2. \]

**Proof.** Let \( S_n \) be a star graph whose vertex set is \( V = \{v_0, v_1, \ldots, v_{n-1}\} \) and the edge set is \( E = \{e_i = v_0v_i/1 \leq i \leq n-1\} \). Let \( ASS(S_n) \) be an arbitrary super subdivision of a star graph \( S_n \). This graph has \((m_1 + m_2 + \cdots + m_{n-1}) + n \) vertices and \(2(m_1 + m_2 + \cdots + m_{n-1})\) edges.

Denote the vertex set of \( ASS(S_n) \) as \( V = \{v_i \cup v_{0i} \cup v_0/1 \leq i \leq n-1\} \) and the edge set \( E = \{v_0v_{0i} \cup v_{0i}v_i/1 \leq i \leq n-1, 1 \leq j \leq m_i\} \).

Define a function \( l : V(ASS(S_n)) \to N \) by \( l(v_i) = 1, l(v_{0i}) = 1 \) and \( l(v_{0i}/j) = 2 \) for \( 1 \leq i \leq n-1, 1 \leq j \leq m_i \) and \( l(v_{(i+1)/n+1}) = 2 \) for \( 1 \leq i \leq n-1, 1 \leq j \leq m_i \).

Thus \( c(u) \neq c(v) \) for any two adjacent vertices in of \( ASS(S_n) \). Hence the arbitrary super subdivision of star graph admits \( d \)-lucky labeling with \( \eta_{dl}(ASS(S_n)) = 2 \). \( \square \)

**Theorem 6.** The arbitrary super subdivision of wheel graph \( W_n \) is a \( d \)-lucky labeled graph and the \( d \)-lucky number is \( \eta_{dl}(ASS(W_n)) = 2 \).

**Proof.** Let \( W_n \) be a wheel graph whose vertex set is \( V = \{v_0, v_1, v_2, \ldots, v_{n-1}\} \) and the edge set is \( E = \{e_i = v_0v_i/1 \leq i \leq n-1\} \cup \{e_{n-1+i} = v_iv_{i+1}/1 \leq i \leq n-2\} \cup \{e_{2n-2} = v_{n-1}v_1\} \).

Let \( ASS(W_n) \) be an arbitrary super subdivision of a wheel graph \( W_n \). This graph has \((m_1 + m_2 + m_3 + \cdots + m_{2n-2}) + n \) vertices and \(2(m_1 + m_2 + m_3 + \cdots + m_{2n-2})\) edges.

Denote the vertex set of \( ASS(W_n) \) as \( V = \{v_0 \cup v_i \cup v_{0i} /1 \leq i \leq n-1, 1 \leq j \leq m_i\} \cup \{v_{0i}/1 \leq i \leq n-2, 1 \leq j \leq m_{n+i-1}\} \cup \{v_{0i}/1 \leq j \leq m_{2n-2}\} \) and the edge set as \( E = \{v_0v_{0i} \cup v_{0i}v_i/1 \leq i \leq n-1, 1 \leq j \leq m_i\} \cup \{v_{0i}/1 \leq i \leq n-2, 1 \leq j \leq m_{n+i-1}\} \cup \{v_{0i}/1 \leq j \leq m_{2n-2}\} \).

Define a function \( l : V(ASS(W_n)) \to N \) by \( l(v_0) = 1, l(v_i) = 1 \) and \( l(v_{0i}) = 2 \) for \( 1 \leq i \leq n-1, 1 \leq j \leq m_i \) and \( l(v_{0i}/j) = 2 \) for \( 1 \leq i \leq n-1, 1 \leq j \leq m_i \) and \( l(v_{i+1}/n+1) = 2 \) for \( 1 \leq i \leq n-1, 1 \leq j \leq m_i \).
Now \( c(u)'s \) are calculated as follows:
\[
\begin{align*}
c(v_0) &= d(v_0) + \sum l(N(v_0)) = 3(m_1 + m_2 + \ldots m_{n-1}) \\
c(v_1) &= d(v_1) + \sum l(N(v_1)) = 3(m_{2n-2} + m_1).
\end{align*}
\]
For \( 2 \leq i \leq n - 1, \)
\[
\begin{align*}
c(v_i) &= d(v_i) + \sum l(N(v_i)) = 3(m_{n+i-1} + m_{n+i-2}) \\
c(v_{(n-1)1}^{(j)}) &= 4 \text{ for } 1 \leq i \leq n - 1, 1 \leq j \leq m_i. \quad c(v_{(i+1)})^{(j)} = 4 \text{ for } 1 \leq i \leq n - 2, 1 \leq j \leq m_{n+i-1} \\
c(v_{(n-1)1}^{(j)}) &= 4 \text{ for } 1 \leq j \leq m_{2n-2}.
\end{align*}
\]
Thus \( c(u) \neq c(v) \) for any two adjacent vertices in \( \text{ASS}(W_n) \).
Hence the arbitrary super subdivision of wheel graph admits \( d \)-lucky labeling with \( \eta_{dl}(\text{ASS}(W_n)) = 2 \).

**Example 7.**

![Diagram of d-lucky labeling of arbitrary super subdivision of wheel graph ASS(W5)](image_url)

**Theorem 8.** The arbitrary super subdivision of corona of \( P_n \) (comb) graph is a \( d \)-lucky labeled graph and the \( d \)-lucky number is \( \eta_{dl}(\text{ASS}(P_n^+)) = 2 \).
Proof. Let $P^+_n$ be a comb graph whose vertex set is $V = \{v_i, u_i/ 0 \leq i \leq n\}$ and the edge set is $E = \{e_{i+1} = v_iv_{i+1}/ 0 \leq i \leq n-1\} \cup \{e_{n+i+1} = v_iu_i/ 0 \leq i \leq n\}$. Let $ASS(P^+_n)$ be an arbitrary super subdivision of a comb graph $P^+_n$. This graph has $(m_1 + m_2 + \cdots + m_{2n+1}) + 2(n + 1)$ vertices and $2(m_1 + m_2 + \cdots + m_{2n+1})$ edges.

Denote the vertex set of $ASS(P^+_n)$ as $V = \{\{v_i \cup u_i/ 0 \leq i \leq n\} \cup \{v^{(j)}_{i(i+1)}/ 0 \leq i \leq n-1, 1 \leq j \leq m_{i+1}\} \cup \{(uv)^{(j)}_i/ 0 \leq i \leq n, 1 \leq j \leq m_{n+i+1}\}\}$ and the edge set as $E = \{v^{(j)}_{i(i+1)} \cup u^{(j)}_{i(i+1)}v_{i+1}/ 0 \leq i \leq n-1, 1 \leq j \leq m_{i+1}\} \cup \{v_i(uw)^{(j)}_i \cup (uv)^{(j)}_i u_i/ 0 \leq i \leq n, 1 \leq j \leq m_{n+i+1}\}.$

Define a function $l : V(ASS(P^+_n)) \to N$ by $l(v_i) = 1$, $l(u_i) = 1$ for $0 \leq i \leq n$, $l(v^{(j)}_{i(i+1)}) = 2$ for $0 \leq i \leq n-1, 1 \leq j \leq m_{i+1}$ and $l(uw)^{(j)}_i = 2$ for $0 \leq i \leq n, 1 \leq j \leq m_{n+i+1}$.

Now $c(u)$'s are calculated as follows:
\[c(v_0) = d(v_0) + \sum l(N(v_0)) = 3(m_1 + m_{n+1});\]
\[c(v_i) = d(v_i) + \sum l(N(v_i)) = 3(m_i + m_{i+1})\text{ for } 1 \leq i \leq n - 1;\]
\[c(v_n) = 3(m_n + m_{2n+1});\]
\[c(u_i) = 3(m_{n+i});\]
\[c(u_i) = 3(m_{n+i})\text{ for } 1 \leq i \leq n;\]
\[c(v^{(j)}_{i(i+1)}) = 4\text{ for } 0 \leq i \leq n-1, 1 \leq j \leq m_{i+1};\]
\[c(uw)^{(j)}_i = 4\text{ for } 0 \leq i \leq n, 1 \leq j \leq m_{n+i+1}.\]

Thus $c(u) \neq c(v)$ for any two adjacent vertices in $ASS(P^+_n)$. Hence the arbitrary super subdivision of comb graph admits $d$-lucky labeling with $d_{1d}(ASS(P^+_n)) = 2$.\hfill \Box

Example 9.

![Figure 4: $d$-Lucky labeling of arbitrary super subdivision of comb graph $ASS(P^+_n)$](image_url)

Figure 4: $d$-Lucky labeling of arbitrary super subdivision of comb graph $ASS(P^+_n)$
3 Conclusion

In this paper we have proved the existence of $d$-lucky labeling of path, cycle, star, wheel and comb graphs and shown that their lucky number is $2$.

References


