A STUDY ON DETOUR RADIAL GRAPH OF CYCLE GRAPHS,
GRAPH PERMUTATION AND INVERSION

1V.MOHANASELVI
Department of Mathematics, Nehru Memorial College
Puthanampatti-621007, Trichy, Tamil Nadu, India.
v.mohanaselvi@gmail.com

2M.SURESH
Department of Mathematics, SRM University
Kattankulathur - 603 203 Tamil Nadu, India.
m.sureshm.sc.mphil@gmail.com

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Abstract
In this paper, the properties of Detour Radial graph of
cycle graphs $DR(C_n)$ are studied, some results on Laplacian
Matrix of $DR(C_n)$ are obtained, also graph permutation and
Inversion are discussed for complete graph.

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Laplacian matrix, Permutation and Inversion.

1 INTRODUCTION

By a graph, we means finite simple and connected graph. The basic
graph theoretical terminology we referred from Harary [8] and the
basic matrix theory is taken from Elementary Matrix Algebra by Franz E. Hohn [11]. The concept of Radial graph was introduced by Kathiresan. K.M in [1] and was further extended by Kathiresan. K. M and Marimuthu.G in [2,3]. Then followed by [1,2,3] we introduced the concept Detour Radial graph in [4]. The advantage of Detour concept is to cover maximum number vertices in a single pass. For instance, a postman needs to deliver the post in his area, Weiner concept is not applicable and hence the Detour concept was introduced in Radial graphs, followed by this the detour radial number and detour diametrical number are defined and studied in [5].

Let \( G \) be any graph, then length of longest path in graph \( G \) is called detour distance between \( u \) and \( v \) and it is denoted by \( D(u,v) \). The Detour eccentricity \( e_D(G) \) of a vertex \( u \) is the distance to a vertex farthest from \( u \). The Detour radius \( r_D(G) \) is the minimum detour eccentricity among the vertices of \( G \) and the detour diameter \( d_D(G) \) is the maximum detour eccentricity among the vertices of \( G \). A graph \( G \) for which \( r_D(G) = d_D(G) \) is called a self-centered graph.

Two vertices of a graph are said to be detour radial to each other if the detour distance between them is equal to the detour radius of the graph. A detour radial graph of a graph \( G \) denoted by \( DR(G) \) and it has the same vertex set as in \( G \) and two vertices are adjacent in \( DR(G) \) if and only if they are detour radial in \( G \).

Let \( v_1, v_2, ..., v_n \) is the set of vertices of \( G \), then the adjacency matrix of \( G \), \( A(G) = A = [a_{ij}] \) is an \( n \)-by-\( n \) matrix, where \( a_{ij} = 1 \) if \( v_i \) and \( v_j \) are adjacent and \( a_{ij} = 0 \) otherwise.

The non-adjacency matrix of \( G \) is defined as \( A_N(G) = A_N = [a_{ij}] \) is an \( n \)-by-\( n \) matrix, where \( a_{ij} = 0 \) if \( v_i \) and \( v_j \) are adjacent and \( a_{ij} = 1 \) otherwise.

The degree matrix of \( G \), \( D(G) \), is an \( n \)-by-\( n \) diagonal matrix such that \( D(G)_{ii} = d_i \) where \( d_i \) is the degree of vertex \( v_i \) (the number of edges incident with \( v_i \)).

The energy of \( G \) was first defined by Gutman in 1978 as the sum of the absolute values of the eigenvalues of \( A(G) \).

\[
E(G) = \sum_{i=1}^{n} |\lambda_i|
\]


2 PRELIMINARIES

2.1 Theorem [4]
Let $C_n$ be any cycle on $n \geq 3$ vertices, then

$$r_D(C_n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n+1}{2}, & \text{if } n \text{ is odd} \end{cases}$$

2.2 Theorem [4]
Let $C_n$ be any cycle on $n \geq 3$ vertices, then

$$DR(C_n) = \begin{cases} \left(\frac{n}{2}\right) P_2, & \text{if } n \text{ is even} \\ \cong C_n, & \text{if } n \text{ is odd} \end{cases}$$

2.3 Theorem [5]
Detour Radial graph of complete tripartite, wheel, cone and double fan graphs are complete graph with same size.

3 PROPERTIES OF DETOUR RADIAL GRAPH FOR CYCLE GRAPH

- Let $C_n$ be the cycle matrix with $n$ vertices and the following are the properties of adjacent matrix for $DR(C_n)$.

(i) $\det(DR(C_n)) = \begin{cases} 1, & \text{if } n \equiv 0(\mod 4) \text{ and } n \geq 4 \\ -1, & \text{if } n \equiv 2(\mod 4) \text{ and } n \geq 4 \\ 2, & \text{if } n \text{ is odd and } n \geq 5 \end{cases}$

(ii) $DR(C_n)$ is symmetric, if $A = A^T$

(iii) $DR(C_n)$ is orthogonal, if $AA^T = A^TA = I$ and $n$ is even

(iv) $DR(C_n)$ is Involutory, if $A^2 = I$ and $n$ is even

(v) Trace is zero

(vi) $(A^T)^T = A$, $(kA)^T = kA^T$, if $A$ is the adjacency matrix of and $k$ is scalar.
(vii) Let $A$ be the adjacency matrix for $DR(C_n)$ then the even powers of the matrix $A$ is identity matrix and odd powers of the matrix $A$ is same.

Let $C_n$ be the cycle matrix with $n$ vertices and the following are the properties of non-adjacent matrix for $DR(C_n)$.

(i) $det(DR(C_n)) = \begin{cases} (1-n), & \text{if } n \equiv 0 \mod{4} \text{ and } n \geq 4 \\ (n-1), & \text{if } n \equiv 2 \mod{4} \text{ and } n \geq 4 \\ (n-2), & \text{if } n \text{ is odd and } n \geq 5 \end{cases}$

(ii) $DR(C_n)$ is symmetric, if $A_N = (A_N)^T$

(iii) The trace of $DR(C_n)$ is $n$

Let $C_n$ be the cycle matrix with $n$ vertices and the following are the properties of degree matrix for $DR(C_n)$.

(i) $det(DR(C_n)) = \begin{cases} 1, & \text{if } n \text{ is even and } n \geq 4 \\ 2^n, & \text{if } n \text{ is odd and } n \geq 5 \end{cases}$

(ii) $DR(C_n)$ is symmetric, if $A = A^T$

(iii) $DR(C_n)$ is orthogonal, if $AA^T = A^TA = I$ and $n$ is even

(iv) $DR(C_n)$ is Involutory , if $A^2 = I$ and $n$ is even

(v) The trace of $DR(C_n)$ is $n$

(vi) $(A^T)^T = A, (kA)^T = kA^T$, if $A$ is the adjacency matrix of and $k$ is scalar.

(vii) Let $A$ be the degree matrix for $DR(C_n)$ then the powers of the matrix $A$ is identity matrix.

4 RESULTS ON LAPLACIAN ENERGY OF CYCLE GRAPH

4.1 Definition

The Laplacian matrix of a graph, $L(G)$, is defined as $L(G) = D(G) - A(G)$; where $D(G)$ is the degree matrix, and $A(G)$ is the adjacency matrix of $G$. 
4.2 Definition

Let $\mu_1, \mu_2, ..., \mu_n$ be the eigenvalues of $L(G)$. Then the Laplacian energy $LE(G)$, is defined as

$$LE(G) = \sum_{i=1}^{n} | \mu_i - \frac{2m}{n} |$$

4.3 Theorem

Let $DR(C_n)$ be Detour Radial of cycle graph with $n$ vertices and $H$ be an induced sub-graph. Suppose $\tilde{H}$ denotes the union of $H$ and vertices of $G - DR(C_n)$ . Then

$$LE(DR(C_n)) - LE(\tilde{H}) \leq LE(DR(C_n)) - E(H) \leq LE(DR(C_n)) + LE(\tilde{H})$$

Where $E(H)$ is the edge set of H.

4.4 Theorem

Suppose $\tilde{H}$ consists of $DR(P_2)$ and $n - 2$ isolated vertices. Then

$$LE(\tilde{H}) = \frac{4(n - 1)}{n}$$

4.5 Theorem

Suppose $H$ is a single edge $e$ and $\tilde{H}$ consists of $e$ and $n - 2$ isolated Vertices. Then

$$LE(DR(C_n)) - \frac{4(n - 1)}{n} \leq LE(DR(C_n)) - \{e\} \leq LE(DR(C_n)) + \frac{4(n - 1)}{n}$$

4.6 Theorem

Let C be a partitioned matrix for Detour Radial graph of Cycle with $n$ vertices, Where $C = \begin{bmatrix} A & X \\ Y & B \end{bmatrix}$ and both $A$ and $B$ are square matrices, we have

$$\sum_j s_j(A) + \sum_j s_j(B) \leq \sum_j s_j(C)$$
4.7 Theorem
Let $C$ be a partitioned matrix for Detour Radial graph of Cycle with $n$ vertices, Where $C = \begin{bmatrix} A & X \\ Y & B \end{bmatrix}$ and both $A$ and $B$ are square matrices, we have
\[ \sum_j s_j(A) \leq \sum_j s_j(C) \]
Equality holds if and only if $X, Y$, and $B$ are all zero matrices.

5 GRAPH PERMUTATIONS AND INVERSIONS

In this section, we give the results of complete graph. From the Theorem [2.2] Detour Radial graph of complete tripartite, wheel, cone and double fan graphs are complete graph with same size. Therefore the following result has the similar proof of Detour Radial graph of complete tripartite, wheel, cone and double fan graphs. Let $j_1, j_2, ..., j_n$ be one of the integers from 1 to $n$ and if each of integers from 1 to $n$ appears once and only once among them $j_i$'s, then we call the set of integers $j_1, j_2, ..., j_n$ a permutation of the integer from 1 to $n$.

Let $v_1, v_2, ..., v_n$ be $n$ vertices of a complete graph, one of the vertices from 1 to $n$ and if each of vertices from 1 to $n$ appears once and only once among them, then we call the set of vertices $v_1, v_2, ..., v_n$ a graph permutation of the vertices from 1 to $n$.

In a graph permutation of the vertices from 1 to $n$, a vertices may precede another smaller vertex. When this occurs, we say that the graph permutation contains an inversion.

The total number of inversions in a graph permutation is definition, found by counting the number of smaller vertices following each vertex of the graph permutation. For example, 614325 has eight inversions since 6 is followed by 1, 4, 3, 2 and 5; 4 is followed by 3 and 2; 3 is followed by 2.

A graph permutation is defined to be even or odd according as the total number of inversion in it is odd or even. If any order has zero inversion then it is even. The above example is even graph permutation since it have eight inversions.
When the two graph permutation have both odd or even, they are said to have the same parity. When the one graph permutation is odd and other is even, they are said to have the opposite parity. The interchange of any two vertices of a graph permutation will be called an adjacent transposition. The interchange of any two vertices of a graph permutation, whether adjacent or not, is called a transpose.

5.1 Theorem

If $K_n$ be a complete graph with $n$ vertices then the graph permutation of Detour path from $v_i$ to $v_j, j = 2, 3, 4... n$ is $(n-1)!$.

**Proof:**

Let $v_1, v_2, ..., v_n$ be $n$ vertices of complete graph $K_n$. The first Detour path of complete graph from $v_1$ to $v[n]$ is $(n-2)!$ path, the second Detour path from $v_i$ to $v_{n-1}$ is $(n-2)!$ path. Continue this process we get $(n-1)$ times of the Detour path from $v_1$ to $v_j$.

Thus, the graph permutation of Detour path from $v_1$ to $v_j = (n-2)! + (n-2)! + ... + (n-2)! = (n-1)(n-2)! = (n-1)!$

5.2 Theorem

If $K_n$ be a complete graph with $n$ vertices then the graph permutation of Detour path from $v_i$ to $v_j, i \neq j$ is $n!$.

**Proof:**

Let $v_1, v_2, ..., v_n$ be $n$ vertices of complete graph $K_n$. By the above theorem, the graph permutation of Detour path from $v_1$ to $v_j$ is $(n-1)!$, the graph permutation of Detour path from $v_2$ to $v_j$ is $(n-1)!$. Proceeding this process for all vertices, then we can get, The Detour path from $v_1$ to $v_j = $ Detour path from $v_1$ to $v_j +$ Detour path from $v_2$ to $v_j + ...$ Detour path from $v_n$ to $v_j = (n-1)! + (n-2)! + ... + (n-1)! = n(n-1)! = n!$

5.3 Theorem

The adjacent transposition changes a given graph permutation in to one of opposite parity.

**Proof:**

This theorem may be illustrated by considering the complete graph.
with four vertices 1, 2, 3 and 4. Then by the above theorem the
graph permutation of Detour path is twenty four. Let us consider
the first graph permutation 1234 is even, second is 1243 is odd,
2143 is even, 2413 is odd and so on.
We prove the theorem by nothing that the two vertex to be in-
terchanged are necessarily unequal and that, after the interchange,
these two vertices still follow or proceed all other vertices in the
graph permutation, just as before. Thus the total number of inver-
sion is either increasing or decreasing by exactly one. Hence, the
adjacent transposition changes a given graph permutation in to one
of opposite parity.

5.4 Theorem
If a graph permutation has \( k \) inversions, then it can be reduced to
the natural order by exactly \( k \) adjacent transposition.
Proof:
By the definition, 51243 has five inversions. It may be reduced to
the natural order by interchanging the vertex 5 successively with
the integer 1, 2, 4 and 3, and then interchanging the 4 and 3.
In the graph permutation we assume that \( k \) inversions exist, let \( n \)
be followed by \( v_n \) smaller vertices, \( n-1 \) be followed by \( v_{n-1} \) smaller
vertices and so on. Then sum of every adjacent transposition we
can get \( k \) adjacent transposition.

5.5 Theorem
An even (odd) graph permutation may be reduced to the natural
order only by an even (odd) number of adjacent transpositions.

5.6 Theorem
A graph permutation is even (odd) iff an even (odd) number of
adjacent transposition is required to reduce it to the natural order.

CONCLUSION
In this paper, we found the properties of Detour Radial graph of
cycle graph \( DR(C_n) \) for adjacent matrix, non-adjacency matrix and
diagonal matrix. Also, results on Laplacian Matrix of $DR(C_n)$ and finally found the graph permutation and Inversion of Detour Radial of complete tripartite, wheel, cone and double fan graphs.

References


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