Fluid Queue Modulated by an $M/M/1/N$ Queue Subject to Multiple Exponential Working Vacation

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Abstract

This paper studies a fluid queueing model driven by an $M/M/1/N$ queue subject to working vacation. The underlying system of differential difference equations that governs the process are solved using Laplace transform and generating function methodologies. Explicit expressions for the joint steady state probabilities of the state of the background queueing model and the content of the buffer are obtained in terms of modified Bessel function of the first kind. Numerical illustrations are added to depict the variations of the state probabilities and the buffer content distributions against varying values of the parameters subject to the stability conditions.

Keywords: Generating Function, Laplace Transform, Steady State Probabilities, Buffer Content Distribution

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1 Introduction

A stochastic fluid flow system is an input-output system where the input is modelled as a continuous fluid that enters and leaves the storage device called a buffer according to randomly varying rates. In these models, a fluid buffer is either filled or depleted or both at rates determined by the current state of the background queueing model. Markov modulated fluid queues are a particular class of fluid models useful for modelling many physical phenomenon and they often allow tractable analysis. In addition, fluid models are often useful as approximate models for certain queueing and inventory systems where the flow consists of discrete entities, but the behaviour of individual is not important to identify the performance analysis. Fluid queueing models find a wide range of applications in the field of wireless communications, transport, storage and computer systems and so on.

Recently, fluid models driven by an $M/M/1$ queue subject to various vacation strategies were analysed in steady state by Mao et al [6] and Wang et al [1]. The work was further extended to the stationary analysis of fluid queues driven by an $M/M/1$ queue with multiple exponential vacation and N policy [4]. Mao

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**2 Model Description**

Consider an $M/M/1$ queueing model with finite capacity $N$. Let the customers arrive according to a Poisson process with parameter $\lambda$. The server provides service to the arriving customer according to an exponential distribution with parameter $\mu_1$ and begins a vacation at the instant when the queue becomes empty. When the vacation duration ends, if there is at least one customer in the queue, the server resumes to work.

Otherwise, the server takes another vacation. This policy is termed as the multiple vacation policy. The vacation times are exponentially distributed with parameter $\theta$. During the vacation period, the server serves the arriving customer at a rate, $\mu_2 < \mu_1$. Let $N(t)$ denotes the number of customers in the system at time $t$. Define

\[
J(t) = \begin{cases} 
1, & \text{if the server is busy} \\
0, & \text{if the server is in vacation}. 
\end{cases}
\]

It is well known the $\{(N(t),J(t)), t \geq 0\}$ is a Quasi Birth and Death (QBD) process with the state space $\Omega = \{(0,0) \cup (k,j), k = 1,2,\cdots N \text{ and } j = 0,1\}$. Let $(N,J)$ denote the stationary random vector of the process $\{N(t), J(t), t \geq 0\}$ and let $\pi_{k,j} = \lim_{t \to \infty} P\{N(t) = k, J(t) = j\}, \ (k,j) \in \Omega$.

**3 Steady State Probabilities**

It is readily seen that the system of equations governing the queueing model under steady state can be written as

\[
\begin{align*}
\mu_1 \pi_{1,1} - \lambda \pi_{0,0} + \mu_2 \pi_{1,0} &= 0, \\
\lambda \pi_{k-1,0} - (\lambda + \mu_2 + \theta) \pi_{k,0} + \mu_2 \pi_{k+1,0} &= 0, \quad k = 1,2,\cdots N - 1, \\
\lambda \pi_{N-1,0} - (\mu_2 + \theta) \pi_{N,0} &= 0, \\
\theta \pi_{1,0} - (\lambda + \mu_1) \pi_{1,1} + \mu_1 \pi_{2,1} &= 0, \\
\theta \pi_{k,0} - (\lambda + \mu_1) \pi_{k,1} + \mu_1 \pi_{k+1,1} + \lambda \pi_{k-1,1} &= 0, \quad k = 2,\cdots N - 1, \\
\theta \pi_{N,0} - \mu_1 \pi_{N,1} + \lambda \pi_{N-1,1} &= 0.
\end{align*}
\]

Define the generating function

\[
G(z) = \sum_{k=0}^{N} \pi_{k,0} z^k \quad \text{and} \quad H(z) = \sum_{k=1}^{N} \pi_{k,1} z^k.
\]
By standard methods, steady state probabilities of \( \pi_{k,1} \) for \( k = 1, 2, \cdots, N \) and \( \pi_{k,0} \) for \( k = 1, 2, 3, \cdots, N \) are given by

\[
\pi_{k,0} = \frac{\pi_{0,0}}{\hat{z}^k} + \frac{\pi_{0,0}([z^*]^N - [z^*]^{N+1})}{[\hat{z}^N(1-\hat{z}) - (z^*)^N(1-z^*)]} \frac{([z^*]^k - \hat{z}^k)}{[(z^*)^k]} \quad \text{for} \quad k = 0, 1, 2, \cdots, N. \quad (7)
\]

\[
\pi_{N,0} = \frac{\pi_{0,0}(z^* - \hat{z})}{[\hat{z}^N(1-\hat{z}) - (z^*)^N(1-z^*)]}.
\]

\[
\pi_{k,1} = \frac{\theta}{(\lambda - \mu_1)} \sum_{j=0}^{k-1} \pi_{j,0} - \frac{\theta \lambda}{\mu_1 (\lambda - \mu_1)} \sum_{j=0}^{k-1} \left( \frac{\lambda}{\mu_1} \right)^{k-1-j} \pi_{j,0} - \frac{\theta}{(\lambda - \mu_1)} \sum_{j=0}^{N} \pi_{j,0} + \frac{\theta \lambda}{\mu_1 (\lambda - \mu_1)} \left( \frac{\lambda}{\mu_1} \right)^{k-1} \sum_{j=0}^{N} \pi_{j,0} \quad \text{for} \quad k = 1, 2, \cdots, N. \quad (9)
\]

where \( \hat{z}, z^* = \frac{(\lambda + \mu_2 + \theta) \pm \sqrt{(\lambda + \mu_2 + \theta)^2 - 4\lambda \mu_2}}{2\lambda} \).

### 4 Analysis of Fluid Queue

This section deals with the stationary analysis of a fluid queue modulated by an \( M/M/1/N \) queueing model subject to multiple exponential working vacation. Let \( C(t) \) be the content of the buffer at time \( t \). Furthermore, it is assumed that the content of the buffer increases at the rate of \( r \) when there are customers in the background queueing model, while the buffer content decreases at the rate \( r_0 \) when the system is empty. The dynamics of the buffer content process is given by

\[
d\frac{dC(t)}{dt} = \begin{cases} 0, & N(t) = 0, C(t) = 0 \\ r_0, & N(t) = 0, C(t) > 0, \\ r, & N(t) > 0. \end{cases}
\]

where \( r_0 < 0 \) and \( r > 0 \). Clearly the 3-dimensional process \( \{(N(t), J(t), C(t)), t \geq 0\} \) represent a fluid queue driven by an \( M/M/1/N \) queue with working vacation subject to the stability condition given by

\[
d = r_0 \pi_{0,0} + r \sum_{k=1}^{N} \pi_{k,0} + r \sum_{k=1}^{N} \pi_{k,1} < 0.
\]

Define the joint probability distribution functions of the Markov process \( \{(N(t), J(t), C(t)), t \geq 0\} \) at time \( t \) as

\[
F_{0,0}(t, x) = \Pr\{N(t) = 0, J(t) = 0, C(t) \leq x\} \quad \text{and} \quad F_{0,j}(t, x) = \Pr\{N(t) = k, J(t) = j, C(t) \leq x\} \quad k = 1, 2, 3, \cdots, N \quad \text{and} \quad j = 0, 1.
\]

When the process \( \{(N(t), J(t), C(t)), t \geq 0\} \) is stable, its stationary random vector is denoted by \( (N, J, C) \). Under steady state conditions, let

\[
F_{k,j}(x) = \lim_{t \to \infty} \Pr\{N(t) = k, J(t) = j, C(t) \leq x\} = \Pr\{N = k, J = j, C \leq x\}, \quad x > 0, (k, j) \in \Omega,
\]
Using standard methods, the system of differential difference equations that governs the process \( \{(N(t), J(t), C(t)), t \geq 0\} \) are given by

\[
\begin{align*}
    r_0 \frac{dF_{0,0}(x)}{dx} & = \mu F_{1,1}(x) - \lambda F_{0,0}(x) + \mu_2 F_{1,0}(x), \\
    r \frac{dF_{k,0}(x)}{dx} & = \lambda F_{k-1,0}(x) - (\lambda + \mu_2 + \theta) F_{k,0}(x) + \mu_2 F_{k+1,0}(x) \quad k = 1, 2, \cdots N - 1, \\
    r \frac{dF_{N,0}(x)}{dx} & = \lambda F_{N-1,0}(x) - (\mu_2 + \theta) F_{N,0}(x), \\
    r \frac{dF_{1,1}(x)}{dx} & = \theta F_{1,0}(x) - (\lambda + \mu_1) F_{1,1}(x) + \mu_1 F_{2,1}(x), \\
    r \frac{dF_{k,1}(x)}{dx} & = \theta F_{k,0}(x) - (\lambda + \mu_1) F_{k,1}(x) + \mu_1 F_{k+1,1}(x) + \lambda F_{k-1,1}(x) \quad k = 2, \cdots N - 1, \\
    r \frac{dF_{N,1}(x)}{dx} & = \theta F_{N,0}(x) - \mu_1 F_{N,1}(x) + \lambda F_{N-1,1}(x).
\end{align*}
\]

with the boundary conditions

\[
F_{0,0}(0) = a \quad \text{and} \quad F_{k,j}(0) = 0, \quad \text{for} \quad (k, j) \in \Omega \setminus (0,0).
\]

The constant \( a \) such that \( 0 < a < 1 \) is an unknown to be determined.

5 Solution Methodology

Let \( \hat{F}_{k,j}(s) \) denote the Laplace transform of \( F_{k,j}(x) \) for all \( (k, j) \in \Omega \). Define the generating function,

\[
Q(z, x) = \sum_{k=1}^{N} F_{k,1}(x)z^k.
\]

By standard methods, the system of difference-differential equations represented by equations (13), (14) and (15) leads to a linear differential equation given by

\[
\frac{\partial Q(z, x)}{\partial x} = \left[ \frac{\lambda z}{r} + \frac{\mu_1}{rz} - \left( \frac{\lambda + \mu_1}{r} \right) \right] Q(z, x) + \frac{\theta}{r} \sum_{k=1}^{N} F_{k,0}(x)z^k + \frac{\lambda}{r} (1 - z)z^N F_{N,1}(x) - \frac{\mu_1}{r} F_{1,1}(x).
\]

Integrating the above equation yields

\[
Q(z, x) = \frac{\theta}{r} \int_{0}^{x} \sum_{k=1}^{N} F_{k,0}(y)z^k e^{-\left( \frac{\lambda + \mu_1}{r} \right) (x - y)} e^{\left( \frac{\lambda z}{r} + \frac{\mu_1}{rz} \right) (x - y)} dy
+ \frac{\lambda}{r} (1 - z)z^N \int_{0}^{x} F_{N,1}(y)e^{-\left( \frac{\lambda + \mu_1}{r} \right) (x - y)} e^{\left( \frac{\lambda z}{r} + \frac{\mu_1}{rz} \right) (x - y)} dy
- \frac{\mu_1}{r} \int_{0}^{x} F_{1,1}(y)e^{-\left( \frac{\lambda + \mu_1}{r} \right) (x - y)} e^{\left( \frac{\lambda z}{r} + \frac{\mu_1}{rz} \right) (x - y)} dy.
\]
Using the bessel function identity, if \( \alpha = 2 \frac{\sqrt{\lambda \mu_1}}{r} \) and \( \beta = \sqrt{\lambda \mu_1} \), then
\[
\exp \left[ \left( \frac{\lambda z}{r} + \frac{\mu_1}{r^2} \right) (x - y) \right] = \sum_{k=-\infty}^{\infty} (\beta z)^k I_k(\alpha(x - y)),
\]
in the equation (16) and comparing the coefficients of \( z^k \) on both sides for \( k = 1, 2, \cdots, N \) leads to
\[
F_{k,1}(x) = \frac{\theta}{r} \int_0^x \sum_{m=1}^N F_{m,0}(y) \beta^{k-m}\lambda^{-m}[I_{k-m}(\alpha(x - y)) - I_{k+m}(\alpha(x - y))] e^{-\left( \frac{\lambda + \mu_1 + \theta}{r} \right)(x - y)} dy
\]
\[
+ \frac{\lambda}{r} \int_0^x F_{N,1}(y) \beta^k e^{-\lambda N}(x - y) e^{-\left( \frac{\lambda + \mu_1}{r} \right)(x - y)} dy
\]
\[
- \frac{\lambda}{r} \int_0^x F_{N,1}(y) \beta^{k-N-1}\lambda^{-N-1}[I_{k-N-1}(\alpha(x - y)) - I_{k+N}(\alpha(x - y))] e^{-\left( \frac{\lambda + \mu_1}{r} \right)(x - y)} dy
\]
(17)

Here, \( F_{k,1}(x) \) for \( k = 1, 2, \cdots, N \) obtained in terms of \( F_{k,0}(x) \) and \( F_{N,1}(x) \). It still remains to determine \( F_{k,0}(x) \) for all \( k \) and \( F_{N,1}(x) \). Toward this end, define the generating function
\[
G(z, x) = \sum_{k=1}^{N} F_{k,0}(x) z^k.
\]

By standard methods, the system of difference-differential equations represented by equations (11) and (12) leads to a linear differential equation given by
\[
\frac{\partial G(z, x)}{\partial x} = \left[ \frac{\lambda z}{r} + \frac{\mu_2}{r^2} + \theta \right] G(z, x) + \frac{\lambda}{r} (1 - z) z^N F_{N,0}(x) - \frac{\mu_2}{r} F_{1,0}(x) + \frac{\lambda}{r} z F_{0,0}(x).
\]

Integrating the above equation yields
\[
G(z, x) = \frac{\lambda z}{r} \int_0^x F_{N,0}(y)e^{-\left( \frac{\lambda + \mu_2 + \theta}{r} \right)(x-y)} \exp \left( \frac{\lambda}{r} + \frac{\mu_2}{r^2} \right)(x - y) dy
\]
\[
- \frac{\lambda z}{r} \int_0^{x+N} F_{N,0}(y)e^{-\left( \frac{\lambda + \mu_2 + \theta}{r} \right)(x-y)} \exp \left( \frac{\lambda}{r} + \frac{\mu_2}{r^2} \right)(x - y) dy
\]
\[
- \frac{\mu_2}{r} \int_0^x F_{1,0}(y)e^{-\left( \frac{\lambda + \mu_2 + \theta}{r} \right)(x-y)} \exp \left( \frac{\lambda}{r} + \frac{\mu_2}{r^2} \right)(x - y) dy
\]
\[
+ \frac{\lambda z}{r} \int_0^x F_{0,0}(y)e^{-\left( \frac{\lambda + \mu_2 + \theta}{r} \right)(x-y)} \exp \left( \frac{\lambda}{r} + \frac{\mu_2}{r^2} \right)(x - y) dy.
\]
(18)

By a similar analysis as before, we get
\[
F_{k,0}(x) = \frac{\lambda}{r} \int_0^x F_{N,0}(y) \beta^{k-N}\alpha^N_1[I_{k-N}(\alpha_1(x - y)) - I_{k+N}(\alpha_1(x - y))] e^{-\left( \frac{x + \mu_2 + \theta}{r} \right)(x - y)} dy
\]
\[
- \frac{\lambda}{r} \int_0^x F_{N,0}(y) \beta^{k-N-1}\alpha^{N-1}_1[I_{k-N-1}(\alpha_1(x - y)) - I_{k+N+1}(\alpha_1(x - y))] e^{-\left( \frac{x + \mu_2 + \theta}{r} \right)(x - y)} dy
\]
\[
+ \frac{\lambda}{r} \int_0^x F_{0,0}(y) \beta^{k-1}\alpha^{N-1}_1[I_{k}(\alpha_1(x - y)) - I_{k+1}(\alpha_1(x - y))] e^{-\left( \frac{x + \mu_2 + \theta}{r} \right)(x - y)} dy.
\]
(19)
which implies $F_{k,0}(x)$ for $k = 1, 2, \cdots, N$ is expressed in terms of $F_{N,0}(x)$ and $F_{0,0}(x)$. To determine $F_{N,0}(x)$, $F_{N,1}(x)$ and $F_{0,0}(x)$, substituting $k = N$ in equation (19) and after simplification leads to
\[
F_{N,0}(x) = h(x) * F_{0,0}(x)
\] (20)

with
\[
h(x) = \frac{\lambda \beta_{N-1}^N}{r} [I_{N-1}(\alpha x) - I_{N+1}(\alpha x)] e^{-\left(\frac{\lambda + \mu x}{r}\right)x} + \sum_{j=0}^{\infty} \delta_1(x)^j,
\]

and
\[
\delta_1(x) = \left\{ \frac{\lambda}{r} [I_0(\alpha x) - I_{2N}(\alpha x)] - \frac{\sqrt{\lambda \mu}}{r} [I_1(\alpha x) - I_{2N+1}(\alpha x)] \right\} e^{-\left(\frac{\lambda + \mu x}{r}\right)x}.
\]

Hence $F_{N,0}(x)$ is expressed in terms of $F_{0,0}(x)$. Taking Laplace transform of the equation (19) leads to
\[
\hat{F}_{k,0}(s) = \hat{\Psi}_k(s) \hat{F}_{0,0}(s)
\] (21)

where
\[
\hat{\Psi}_k(s) = \frac{\lambda \beta_{k-N}^N}{r} h(s) \beta_1^{k-N} \left[ \frac{\alpha_1^{-(k-N)} [q - \sqrt{q^2 - \alpha_1^2}]^{k-N}}{q^2 - \alpha_1^2} - \frac{\alpha_1^{-(k+1)} [q - \sqrt{q^2 - \alpha_1^2}]^{k+1}}{q^2 - \alpha_1^2} \right] \]
\[+ \frac{\lambda \beta_{k-1}^{k-1}}{r} \left[ \frac{\alpha_1^{-(k-1)} [q - \sqrt{q^2 - \alpha_1^2}]^{k-1}}{q^2 - \alpha_1^2} - \frac{\alpha_1^{-(k+1)} [q - \sqrt{q^2 - \alpha_1^2}]^{k+1}}{q^2 - \alpha_1^2} \right] = \frac{\lambda}{r} h(s) \beta_1^{k-N-1}
\]

for $k = 1, 2, \cdots, N$. Inversion of equation (21) leads to
\[
F_{k,0}(x) = \Psi_k(x) * F_{0,0}(x), \quad \text{for} \ k = 1, 2, \cdots, N
\] (22)

with
\[
\Psi_k(x) = \frac{\lambda \beta_{k-N}^N}{r} h(x) * [I_{k-N}(\alpha x) - I_{k+N}(\alpha x)] e^{-\left(\frac{\lambda + \mu x}{r}\right)x} + \frac{\lambda \beta_{k-1}^{k-1}}{r} [I_{k-1}(\alpha x) - I_{k+1}(\alpha x)] e^{-\left(\frac{\lambda + \mu x}{r}\right)x}.
\]

Therefore $F_{k,0}(x)$ for all $k$ is expressed in terms of $F_{0,0}(x)$. It still remains to find $F_{N,1}(x)$ and $F_{0,0}(x)$ explicitly. Substituting $k = N$ in equation (17) and after simplification leads to
\[
F_{N,1}(x) = R(x) * F_{0,0}(x).
\] (23)

with
\[
R(x) = \frac{\theta}{r} \sum_{m=1}^{N} \beta^N_m \Psi_m(x) * [I_{N-m}(\alpha x) - I_{N+m}(\alpha x)] e^{-\left(\frac{\lambda + \mu x}{r}\right)x} + \sum_{j=0}^{\infty} \delta(x)^j,
\]

and
\[
\delta(x) = \left\{ \frac{\lambda}{r} [I_0(\alpha x) - I_{2N}(\alpha x)] - \frac{\sqrt{\lambda \mu}}{r} [I_1(\alpha x) - I_{2N+1}(\alpha x)] \right\} e^{-\left(\frac{\lambda + \mu x}{r}\right)x}.
\]
Thus $F_{N,1}(x)$ is expressed in terms of $F_{0,0}(x)$. Finally, we determine $F_{0,0}(x)$ from equation (10). Now, substituting $k = 1$ in equation (17) yields

$$F_{1,1}(x) = g(x) * F_{0,0}(x),$$

(24)

with

$$g(x) = \frac{\theta}{r} \sum_{m=1}^{N} \beta_{1-m} \Psi_{m}(x) * [I_{1-m}(\alpha x) - I_{1+m}(\alpha x)]e^{-(\frac{\lambda}{\beta} + \mu) x} + \frac{\lambda}{r} \beta_{1-N} R(x) * [I_{1-N}(\alpha x) - I_{1+N}(\alpha x)]e^{-(\frac{\lambda}{\beta} + \mu) x}.$$  

(25)

Taking Laplace transform of equation (10) leads to

$$(sr_0 + \lambda)\hat{F}_{0,0}(s) = ar_0 + \mu_1 \hat{F}_{1,1}(s) + \mu_2 \hat{F}_{1,0}(s)$$

which on inversion yields

$$F_{0,0}(x) = a \sum_{i=0}^{\infty} \frac{x^i}{i!r^{i-1}} e^{-r_0 x} * [\mu_1 g(x) + \mu_2 \Psi_1(x)]^{*i}$$

(26)

with

$$\Psi_1(x) = \frac{\lambda \beta_{1-N} \alpha x}{r} h(x) * [I_{1-N}(\alpha x) - I_{1+N}(\alpha x)]e^{-(\frac{\lambda}{\beta} + \mu) x} + \frac{\lambda}{r} I_0(\alpha x) - I_2(\alpha x)]e^{-(\frac{\lambda}{\beta} + \mu) x}.$$  

and $g(x)$ is given in the equation (25). Therefore $F_{k,0}(x)$, $F_{k,1}(x)$, $F_{N,0}(x)$ and $F_{N,1}(x)$ are all expressed interms of $F_{0,0}(x)$ and $F_{0,0}(x)$ is given by (26). Thus all the joint steady state probabilities of the number in the background queueing model and the content of the fluid buffer are explicitly determined using generating function and Laplace transforms in terms of modified Bessel function of first kind. To determine the constant $\alpha$ which represents $F_{0,0}(0)$, adding all the equations (10) to (15) and integrating from zero to infinity gives

$$F_{0,0}(0) = a = \frac{(r_0 - r)\pi_{0,0} + r}{r_0},$$

(27)

where $\pi_{0,0}$ is given by (7). Having determined a closed form expression for all the joint steady state probabilities, the buffer content distribution can be explicitly obtained.

6 Observations

Remark 1: When $N \to \infty$ and the working vacation parameter $\mu_2 = 0$, the model under consideration reduces to a fluid queue driven by an $M/M/1$ queue subject to multiple exponential vacation. Under this assumption, we get

$$F_{k,1}(x) = \frac{\theta}{r} \int_{0}^{x} \sum_{m=1}^{\infty} F_{m,0}(y) \beta_{k-m} [I_{k-m}(\alpha x - y) - I_{k+m}(\alpha x - y)] e^{-(\frac{\lambda}{\beta} + \mu) (x - y)} dy.$$  

(28)

$$F_{k,0}(x) = \left(\frac{\lambda}{r}\right)^k \int_{0}^{x} \frac{(x - y)^{k-1}}{(k-1)!} e^{-(\frac{\lambda}{\beta} + \mu) (x - y)} F_{0,0}(y) dy \quad k = 1, 2, \ldots$$

(29)
Figure 1: Variations of $F_{k,0}(x)$ against $x$

$$F_{0,0}(x) = a \sum_{i=0}^{\infty} \left( \frac{\theta}{r_0} \right)^i \frac{x^k}{k!} e^{-\left( \frac{\lambda}{\mu} \right)x} \left[ \sum_{m=1}^{\infty} \left( \frac{\lambda}{r_\beta} \right)^m \frac{x^{m-1}}{(m-1)!} e^{-\left( \frac{\lambda+1}{\mu} \right)x} \frac{m! \alpha^m x^{-\alpha}}{\alpha^{2m}} e^{-\left( \frac{\lambda+1}{\mu} \right)x} \right]$$ (30)

It is observed that with $r_0 = \sigma_0$ and $r = \sigma$, the above equations (28),(29) and (30) coincides with the equations (4.6), (4.9) and (4.11) respectively, of Sherif I.Ammar [5].

**Remark 2:** When $N \to \infty$ and the working vacation parameter $\mu_2 = 0$, the model also reduces to fluid queue driven by an $M/M/1$ with multiple exponential vacation and $N$-policy with $N = 1$. With $\frac{\nu - \sqrt{\nu^2 - \sigma^2}}{2} = \frac{\mu r_0(s)}{\nu}$, where $r_0(s) = \frac{\nu + \alpha + \beta + \mu}{\nu - \theta}$, leads to

$$\hat{F}_{0,0}(s) = \frac{\alpha r_0 \left[ sr + \lambda + \theta - \mu r_0(s) \right]}{(sr + \lambda)(sr + \lambda + \theta - \mu r_0(s)) - \mu r_0(s)}$$

It is observed that with $r_0 = \sigma_0$ and $r = \sigma$, the above result coincides with $\hat{F}_0(s)$ presented in Remark 1 of page 128 in Mao *etal* [4].

### 7 Numerical Illustrations

This section illustrates the variations of the joint steady state probabilities of the buffer content and the state of the background queueing model for varying values of the parameters. Figure 2 depicts the variations of $F_{k,0}(x)$ for $k = 0, 1, 2, 3$ against the buffer content, $x$ for $\lambda = 1, \mu_1 = 2, \mu_2 = 1.2, \theta = 0.5, r_0 = -2.5, r = 1$ and $N = 50$. As discussed earlier, the boundary conditions are given by $F_{0,0}(0) = a = 0.1180$ and $F_{k,0}(0) = 0$, for $(k, j) \in \Omega \setminus (0, 0)$. Further as $x$ tends to infinity, $F_{k,0}(x)$ will converge to the corresponding steady state probabilities of the background queueing model, $\pi_{k,0}$ (given by equation (7)). Therefore $F_{0,0}(x)$ starts with $a = 0.1180$ and converge to $\pi_{0,0} = 0.3700$ as $x$ increases. Similarly $F_{k,0}(x)$ starts with zero, increases with increase in $x$ and converges to $\pi_{k,0}$ as $x$ tends to infinity. Figure 3 presents the corresponding behavior of $F_{1,1}(x)$ against $x$ for the same set of parameter values. As before, $F_{k,1}(x)$ for $k = 1, 2, 3, 4$, increases with increase in $x$ and tends to the value of $\pi_{k,1}$ (given by equation (9)) as $x$ tends to infinity. Figure 4 depicts the behavior of the buffer content distribution, $F(x)$ against $x$ for $\lambda = 1, \mu_1 = 2, \mu_2 = 1.2, r_0 = -2.5, r = 1, N = 50$ and varying value of $\theta$. It is seen that $F(x)$ increases with increase in the value of the parameter $x$ and converges to 1 as $x$ tends to infinity. Thus all the joint state probabilities are explicitly obtained under steady state and their corresponding behaviour is illustrated numerically for varying values of the parameters.
Figure 2: Variations of $F_{k,1}(x)$ against $x$

Figure 3: Variations of buffer content distribution against $x$ for different values of $\theta$

References


