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# ON SOME COEFFICIENT BOUNDS OF JANOWSKI $\lambda$-SPIRALLIKE MULTIVALENT FUNCTIONS

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Abstract. In the present paper, we obtain sharp bounds for the Fekete-Szegö coefficient of Janowski $\lambda$-spirallike functions with the $k^{th}$ root transform of multivalent functions.

## 1. Introduction

Let $\mathcal{A}_p$ denote the class of all analytic functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, p \in \mathbb{N} \tag{1}$$

defined on the open unit disk

$$\mathbb{U} = \{z \ : \ z \in \mathbb{C} \ : \ |z| < 1\}, \tag{2}$$

and note that $\mathcal{A}_1 := \mathcal{A}$. For $f(z)$ given by (1) and $g(z)$ given by

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n, \tag{3}$$

their convolution (or Hadamard product), denoted by $(f * g)$, is defined as

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n. \tag{4}$$

With a view to recalling the principle of subordination between analytic functions, let the functions $f$ and $g$ be analytic in $\mathbb{U}$. Then we say that the function $f$ subordinate $g$ if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with

$$w(0) = 0, \qquad |w(z)| < 1 \qquad (z \in \mathbb{U}), \tag{5}$$

such that

$$f(z) = g(w(z)) \qquad (z \in \mathbb{U}). \tag{6}$$

We denote this subordination by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z), \qquad (z \in \mathbb{U}). \tag{7}$$

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In particular, if the function $g$ is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0), \qquad f(\mathbb{U}) \subset g(\mathbb{U}). \tag{8}$$

For $-1 \leq B < A \leq 1$, let $P(A, B)$ [5] denote the class of functions which are of the form

$$\phi_1(z) = \frac{1 + Az}{1 + Bz}. \tag{9}$$

Let $P(A, B, p, \alpha)$ denote the class of functions

$$\phi(z) = p + \sum_{n=1}^{\infty} B_n z^n$$

which are analytic in $\mathbb{U}$ and

$$\phi(z) = (p - \alpha)\phi_1(z) + \alpha, \qquad \phi_1(z) \in P(A, B). \tag{10}$$

Using (9) and (10), one can show that $\phi(z) \in P(A, B, p, \alpha)$ if and only if

$$\phi(z) = \frac{p + \gamma z}{1 + Bz}, \qquad z \in \mathbb{U} \tag{11}$$

where $\gamma = (p - \alpha)A + \alpha B$.

Let $S_p^{\lambda}(A, B, b)$ denote the family of functions $f(z) \in \mathcal{A}_p$ such that $f(z) \in S_p^{\lambda}(A, B, b)$ if and only if

$$p + \frac{e^{i\lambda}}{b \cos \lambda} \left( \frac{zf'(z)}{f(z)} - p \right) \prec \phi(z), \qquad \text{for} \qquad \phi(z) \in P(A, B, p, \alpha). \tag{12}$$

for some real $\lambda$, $|\lambda| < \frac{\pi}{2}$, $b \neq 0$ complex number. The class $S_p^{\lambda}(A, B, b)$ is called Janowski $\lambda$-spirallike multivalent functions with complex order introduced and studied by Dileep and Latha [2].There are various subclass of spiral-like function related to the study of Geometric Function Theory, one of these class of spiral-like function in multivalent function which generalized and studied by Ramachandran et al. [8](also see [9],[11] and [10]).By specializing $A$, $B$, $b$, $\lambda$ and $\alpha$ in the function class $S_p^{\lambda}(A, B, b)$, we obtain several well-known as well as new subclasses of analytic functions listed below:

(1) If we take $\lambda = 0$ and $b = 1$, then

$$S_p^0(A, B, 1) \equiv S(A, B, p, \alpha) = \left\{ f : f \in \mathcal{A}_p, \quad \frac{zf'(z)}{f(z)} \prec \phi(z). \quad z \in \mathbb{U} \right\}$$

The class $S(A, B, p, \alpha)$ has been introduced and studied by M.K.Aouf in [1].

(2) If we take $\lambda = 0$, $b = 1$, $\alpha = 0$ and $p = 1$ then the class $S_p^{\lambda}(A, B, b)$ reduces to the class

$$S_1^0(A, B, 1) \equiv S^*(A, B) = \left\{ f : f \in \mathcal{A}, \quad \frac{zf'(z)}{f(z)} \prec \phi(z). \quad z \in \mathbb{U} \right\}$$

The class $S^*(A, B)$ has been introduced and studied by Janowski in [5].

(3) If we take $A = 1$, $B = -1$ $\lambda = 0$ and $b = 1$, then the class $S_p^\lambda(A, B, b)$ reduces to the class

$$S_\alpha(p) = \left\{ f : f \in \mathcal{A}_p, \quad \frac{zf'(z)}{f(z)} \prec \phi(z). \quad z \in \mathbb{U} \right\}$$

The class $S_\alpha(p)$ has been introduced and studied by Golzina in [4].

(4) If we take $\lambda = 0$, $b = 1$ and $\alpha = 0$ then the class $S_p^\lambda(A, B, b)$ reduces to the class

$$S_p(A, B) = \left\{ f : f \in \mathcal{A}_p, \quad \frac{zf'(z)}{f(z)} \prec \phi(z). \quad z \in \mathbb{U} \right\}$$

The class $S_p(A, B)$ has been introduced and studied by Ganesan in [3].

(5) If we take $\lambda = 0$, $B = 0$, $\alpha = 0$ and $p = 1$ then the class $S_p^\lambda(A, B, b)$ reduces to the class

$$S(A, b) = \left\{ f : f \in \mathcal{A}, \quad \frac{zf'(z)}{f(z)} \prec \phi(z). \quad z \in \mathbb{U} \right\}$$

The class $S(A, b)$ has been introduced and studied by Silverman in [12].

(6) If we take $p = 1$, then the class $S_p^\lambda(A, B, b)$ reduces to the class

$$S^\lambda(A, B, b) = \left\{ f : f \in \mathcal{A}_p, \ 1 + \frac{e^{i\lambda}}{b \cos \lambda} \left( \frac{zf'(z)}{f(z)} - 1 \right) \prec \phi(z). \ z \in \mathbb{U} \right\}$$

The class $S^\lambda(A, B, b)$ has been introduced and studied by Polatoglu et al. in [7].

## 2. Some preliminaries

In the present sequel, we estimate a sharp bound in connection with first three coefficients in Theorem 3 for the more generalized class $S_p^\lambda(A, B, b)$ under the $k^{th}$ root transformation.

Let $\Omega$ be the class of analytic functions of the form

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots \tag{13}$$

in the open unit disk $\mathbb{U}$ satisfying $|w(z)| < 1$.

To prove our main result, we need the following.

**Lemma 1.** [6] *If $w \in \Omega$, then for any complex number $t$,*

$$\left| w_2 - t w_1^2 \right| \leq \max\{1; |t|\}. \tag{14}$$

*The result is sharp for the functions $w(z) = z$ or $w(z) = z^2$.*

**Definition 2.** For $f(z) \in \mathcal{A}_p$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n,$$

the $k^{th}$ root transformation is defined by

$$F(z) = \left[f(z^k)\right]^{\frac{1}{k}} = z^p + \sum_{n=1}^{\infty} b_{p+nk}\ z^{p+nk} \tag{15}$$

## 3. Coefficient bounds

By making use of the Lemma 1, we prove the following bound for the class $S_p^{\lambda}(A, B, b)$.

**Theorem 3.** *Let $p(z) = p + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, and if $f(z)$ given by (1) belongs to $S_p^{\lambda}(A, B, b)$, then for any complex number $\mu$,*

$$\left|b_{p+2k} - \mu b_{p+k}^2\right| \leq \frac{pB_1 e^{-i\lambda} b \cos\lambda}{2k} \max\left\{1, \left|pB_1\Lambda(\mu, k, \lambda, b) - \frac{B_2}{B_1}\right|\right\} \tag{16}$$

*where*

$$\Lambda(\mu, k, \lambda, b) := \left[\left(\frac{2\mu + k - 1}{k}\right) be^{-i\lambda}\cos\lambda - 1\right]$$

*This result is sharp.*

*Proof.* If $f(z) \in S_p^{\lambda}(A, B, b)$, then there is a schwarz function $w(z) \in \Omega$ of the form (13) such that

$$p + \frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{zf'(z)}{f(z)} - p\right) = p(w(z)) \tag{17}$$

Since

$$\frac{zf'(z)}{f(z)} = p + a_{p+1}z + (2a_{p+2} - a_{p+1}^2)z^2 + (3a_{p+3} - 3a_{p+1}a_{p+2} + a_{p+1}^3)z^3 + \ldots$$

we have from (17),

$$a_{p+1} = (pB_1 w_1)e^{-i\lambda} b\cos\lambda, \tag{18}$$

$$a_{p+2} = \frac{1}{2}\left\{pB_1 w_2 + p\left(B_2 + pB_1^2\right)w_1^2\right\} e^{-i\lambda} b\cos\lambda, \tag{19}$$

and

$$a_{p+3} = \frac{pB_1}{3}\left\{w_3 + \left(\frac{4B_2 + 3pB_1^2}{2B_1}\right)w_1 w_2 + \left(\frac{2B_3 + 3pB_1B_2 + p^2B_1^3}{2B_1}\right)w_1^3\right\} e^{-i\lambda} b\cos\lambda, \tag{20}$$

For a function $f(z)$ given by (1), we have

$$\begin{aligned}\left[f\left(z^k\right)\right]^{\frac{1}{k}} &= z^p + \left(\frac{1}{k}a_{p+1}\right)z^{p+k} + \left(\frac{1}{k}a_{p+2} - \frac{k-1}{2k^2}a_{p+1}^2\right)z^{p+2k} \\ &\quad + \left(\frac{1}{k}a_{p+3} - \frac{k-1}{k^2}a_{p+2}a_{p+1} + \frac{2k^2 - 3k + 1}{6k^3}a_{p+1}^3\right)z^{p+3k} + \ldots\end{aligned} \tag{21}$$

The equations (15) and (21) yields

$$b_{p+k} = \frac{1}{k}a_{p+1} \tag{22}$$

$$b_{p+2k} = \frac{1}{k}a_{p+2} - \frac{k-1}{2k^2}a_{p+1}^2 \tag{23}$$

$$b_{p+3k} = \frac{1}{k}a_{p+3} - \frac{k-1}{k^2}a_{p+2}a_{p+1} + \frac{2k^2-3k+1}{6k^3}a_{p+1}^3 \tag{24}$$

On substituting for $a_{p+1}$, $a_{p+2}$ and $a_{p+3}$ in (22), (23) and (24), it follows

$$\begin{aligned} b_{p+k} &= \frac{1}{k}(pB_1w_1)e^{-i\lambda}b\cos\lambda \\ b_{p+2k} &= \frac{1}{k}\frac{1}{2}\left\{pB_1w_2 + p\left(B_2 + pB_1^2\right)w_1^2\right\}e^{-i\lambda}b\cos\lambda - \frac{k-1}{2k^2}a_{p+1}^2 \\ b_{p+3k} &= \frac{1}{k}a_{p+3} - \frac{k-1}{k^2}a_{p+2}a_{p+1} + \frac{2k^2-3k+1}{6k^3}a_{p+1}^3 \end{aligned}$$

and hence

$$b_{p+2k} - \mu b_{p+k}^2 = \frac{pB_1e^{-i\lambda}b\cos\lambda}{2k}\left[w_2 - \nu w_1^2\right]$$

where $\nu = \left[\left(\frac{2\mu+k-1}{k}\right)pB_1e^{-i\lambda}b\cos\lambda - \frac{B_2}{B_1} - pB_1\right]$

The results (16) is established by an application of Lemma 1.

The result is sharp for the function defined by

$$p + \frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{zf'(z)}{f(z)} - p\right) = p(z^2)$$

and

$$p + \frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{zf'(z)}{f(z)} - p\right) = p(z)$$

□

If $k = 1$

**Corollary 4.** *Let $p(z) = p + B_1z + B_2z^2 + B_3z^3 + \cdots$, and if $f(z)$ given by (1) belongs to $S_p^\lambda(A, B, b)$, then for any complex number $\mu$,*

$$\left|b_{p+2} - \mu b_{p+1}^2\right| \leq \frac{pB_1e^{-i\lambda}b\cos\lambda}{2}\max\left\{1, \left|pB_1\Lambda_1(\mu,\lambda,b) - \frac{B_2}{B_1}\right|\right\} \tag{25}$$

*where*

$$\Lambda_1(\mu,\lambda,b) := \left[2\mu be^{-i\lambda}\cos\lambda - 1\right]$$

*This result is sharp.*

If $p = 1$

**Corollary 5.** *Let $p(z) = p + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, and if $f(z)$ given by (1) belongs to $S^{\lambda}(A, B, b)$, then for any complex number $\mu$,*

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \leq \frac{B_1 e^{-i\lambda} b \cos\lambda}{2k} \max\left\{1, \left|B_1 \Lambda_2(\mu, k, \lambda, b) - \frac{B_2}{B_1}\right|\right\} \tag{26}$$

*where*

$$\Lambda_2(\mu, k, \lambda, b) := \left[\left(\frac{2\mu + k - 1}{k}\right) b e^{-i\lambda} \cos\lambda - 1\right]$$

*This result is sharp.*

If $p = 1$, $k = 1$

**Corollary 6.** *Let $p(z) = p + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, and if $f(z)$ given by (1) belongs to $S^{\lambda}(A, B, b)$, then for any complex number $\mu$,*

$$\left|b_3 - \mu b_2^2\right| \leq \frac{B_1 e^{-i\lambda} b \cos\lambda}{2} \max\left\{1, \left|B_1 \Lambda_3(\mu, k, \lambda, b) - \frac{B_2}{B_1}\right|\right\} \tag{27}$$

*where*

$$\Lambda_3(\mu, k, \lambda, b) := \left[2\mu b e^{-i\lambda} \cos\lambda - 1\right]$$

*This result is sharp.*

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