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A REVIEW OF FUZZY GRAPH THEORY

V. MYTHILI, M. KALIYAPPAN, AND S. HARIHARAN

Abstract. Fuzzy graph theory has wider range of applications in logic, algebra, topology, operations research, pattern recognition, artificial intelligence, neural networks and several other fields. In this article, recent developments in fuzzy graph theory have been reviewed.

1. Introduction

Graph theory serves as a mathematical model to represent any system having a binary relation and Fuzzy set originated in a seminal paper presented in 1965 by Zadeh [57]. Fuzzy graph theory, a combination of graph theory and fuzzy set theory have been applied in various fields of science and engineering. Rosenfeld [46] considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs in 1975. The book entitled Fuzzy graphs and Fuzzy hypergraphs written by Mordeson et al. [31] is an excellent source for research in fuzzy graphs and fuzzy hypergraphs. Sunitha et al. [56] and Animesh Kumar Sharma et al. [6] reviewed the works of fuzzy graphs. This work extends the survey results in recent times.

2. FUZZY GRAPHS

In 1973, Kaufmann defined fuzzy graphs for the first time. Then Azriel Rosenfeld developed the theory of fuzzy graphs in 1975. Here we have presented the contributions of several authors towards the field of this fast growing fuzzy graph theory.

Definition 1. A fuzzy graph $G = (\sigma, \mu)$ is a pair of functions $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ such that $\mu(u, v) \leq \sigma(u) \cap \sigma(v)$ for all u, v in V where V is the vertex set. The fuzzy graph $H = (\tau, \rho)$ is called a fuzzy subgraph of $G = (\sigma, \mu)$ if $\tau(u) \leq \sigma(u)$ for all

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$u \in V$ and $\rho(u, v)(u, v)$ for all $u, v \in V$. Also H is called a spanning subgraph if $\tau(u) = \sigma(u)$ for all $u \in V$.

2.1. **Properties of fuzzy graphs.** Mcallister [25] proved that intersection of two fuzzy graphs is again a fuzzy graph. Radha et al. [45] explained the degree of an edge in union and join of fuzzy graph through some illustrations. Nagoor Gani et al. [36] discussed degree of a vertex in composition and Cartesian product. Nirmala et al. [40] explained degree of a vertex in Tensor product and normal product of fuzzy graph. Prabir Bhattacharya et al. [44] presented an algorithm to find the supremum of max-min powers of a map by characterizing the path in a fuzzy graph. Moderson et al. [29] proved a necessary and sufficient condition for a graph to be a Cartesian product of two fuzzy subgraphs and union of two fuzzy subgraphs of a graph is again a fuzzy subgraph. Nair [39] discussed few properties of complete fuzzy graph and fuzzy trees. He proved triangle laws, parallelogram laws and few equivalences of bridges in fuzzy graphs. Sunitha et al. [55] studied certain properties of fuzzy bridges, fuzzy cut nodes and using them they obtained a characterization of fuzzy trees and fuzzy cut node. Nagoor Gani et al. [33] proved the inequality involving order and size of a fuzzy graph. Mordeson [29] has defined the complement of a fuzzy graph. Arindam Dey et al. [7] presented a program to determine the fuzzy complement of a fuzzy graph. Nagoor Gani and Chandrasekaran [34] defined μ-complement of a fuzzy graph. Nagoor Gani et al. [35] discussed few properties of μ-complement of fuzzy graph. They proved that μ-complement of a fuzzy graph has isolated nodes if and only if the given graph is a strong fuzzy graph. Sandeep Narayanan et al. [49] presented an illustration of a fuzzy graph containing three vertices and its complement.

2.2. **Strong arcs in fuzzy graphs.** Bhutani et al. [10] defined strong arcs in fuzzy graphs and proved that a strong arc need not be a bridge whereas a bridge is a strong arc. Sunil Mathew et al. [53] proved that a fuzzy graph is a fuzzy tree iff there exists an unique α-strong path between any two nodes and also proved that an arc in a fuzzy tree is α-strong iff it is an arc of the spanning tree of the fuzzy graph.

2.3. **Connectivity in a fuzzy graph.** Sandeep Narayanan et al. [49] proved that the complement of a fuzzy graph is connected if the given fuzzygraph without m-strong arcs is connected. Also proved that a graph and its complement are connected iff the given graph has atleast one connected spanning fuzzy subgraph without any m-strong arcs.

2.4. **Blocks and Cycles in fuzzy graphs.** Mini Tom et al. [26] proved that the fuzzy graph satisfying the condition that either (u, v) is an δ-arc or $\mu(u, v) = 0$ is a block iff there exists atleast two internally disjoint strongest $u - v$ paths. Also they proved that if the underlying graph $G*$ is a complete graph then the fuzzy graph G without δ-arc is a block. Mordeson et al. [30] proved that a fuzzy graph which is a cycle is a fuzzy cycle iff it is not a fuzzy tree. They also proved that the fuzzy graph (σ, μ) does not have a fuzzy bridge iff it is a cycle and μ is a constant function assuming that the dimension of the cycle space of the underlying graph $(\sigma*, \mu*)$ is unity.

2.5. **Domination in fuzzy graphs.** Manjunisha et al. [24] proved that the strong domination number of a non trivial fuzzy graph is equal to the size of the fuzzy graph iff each node is either an isolated node or has an unique strong neighbour and all arcs are strong.

2.6. **Automorphism of fuzzy graphs.** Bhattacharya [9] obtained a fuzzy analog from graph theory to fuzzy graph theory which states that we can associate a group with fuzzy graph as an automorphism group. Bhutani [11] introduced the concept of isomorphism between fuzzy graphs. He proved that every fuzzy group has an embedding into the fuzzy group of the group of automorphism of a fuzzy graph. Let (σ_1, μ_1) and (σ_2, μ_2) be fuzzy subgraphs of graphs (σ_1*, μ_1*) and (σ_2*, μ_2*) respectively. Then Mordeson[28] proved that any weak isomorphism of (σ_1, μ_1) onto (σ_2, μ_2) is again an isomorphism of (σ_1*, μ_1*) onto (σ_2*, μ_2*). Sunitha et al. [54] proved that the set all automorphisms of a fuzzy graph will be a group when the binary relation is set theoretic composition of maps. Sathyaseelan et al. [50] proved that the order and size of any two isomorphic fuzzy graphs are the same. They also proved that the relation Isomorphism between fuzzy graphs satisfies reflexivity, symmetricity and transitivity. i.e. It is an equivalence relation.

2.7. **Coloring and Clustering of fuzzy graphs.** Eslahchi et al. [15] introduced fuzzy coloring of a fuzzy graph. Arindam Dey et.al [8] introduced an algorithm with an illustration to color the complement of a fuzzy graph through α-cuts by considering three cases.

Nivethana et al. [41] presented executive committee problem as an illustration to find the chromatic number of a fuzzy graph. Ananthanarayanan et al. [5] explained how to find the chromatic number of a fuzzy graph using α-cuts by considering fuzzy graphs with crisp vertices and fuzzy edges through illustrations. Sameena [47] presented an algorithm for constructing ϵ-clusters using strong arcs and explained the procedure to obtain ϵ-clusters through some illustrations where $0 \leq \epsilon \leq 1$.

2.8. **Interval valued Fuzzy line graphs.** Moderson [28] presented a necessary and sufficient condition for a fuzzy graph to be a fuzzy line graph. Craine [13] analysed various properties of fuzzy interval graphs. Naga Maruthi Kumari et.al [32] proved that the composition of two strong interval valued fuzzy graphs is a strong interval valued fuzzy graph. Hossein Rashmanlou et.al [19] proved that the semi strong product and strong product of two interval valued fuzzy graphs is complete. Akram [1] stated a proposition that Interval valued fuzzy graph is isomorphic to an interval valued fuzzy intersection graph. Sen et al. [51] proved that fuzzy intersection graph is chordal iff for a, b, c, d in the semigroup, some pair from $\{a, b, c, d\}$ has a right common multiple property.

2.9. **Intuitionistic Fuzzy Graphs.** Deng-Feng li [14] proposed two linear dissimilarity measures between intuitionistic fuzzy sets. Akram et.al [2] discussed few metric aspects of intuitionistic fuzzy graphs. Nagoor Gani et.al [37] proved that the sum of the degree of membership value of all vertices in an intuitionistic fuzzy graph is two times the sum of the membership value of all edges and the sum of the degree of non membership value of all vertices in an intuitionistc fuzzy graph is two times the sum of the non membership value of all edges. Karunambigai et.al[20] discussed three cases where strong path is a strongest path in intuitionistic fuzzy graphs. Akram et.al [3] proved that the join of two strong intuitionistic fuzzy graphs is again a strong intuitionistic fuzzy graph. Karunambigai et.al [21] proved that the order and size of two isomorphic intuitionistic fuzzy graphs are same. Karunambigai et.al [22] proved that every complete intuitionistic fuzzy graph is balanced. Karunambigai et al. [23] presented a necessary and sufficient condition for intuitionistic fuzzy graph to be self centered. Parvathi et al. [43] proved a necessary and sufficient condition for a dominating set to be a minimal dominating set in intuitionistic fuzzy graph. Anthony Shannon et al. [4] presented some generalizations of intuitionistic fuzzy graph. Mishra et al. [27] investigated the results based on product of interval valued intuitionistic fuzzy graph. Sankar Sahoo et al. [48] explained different types of products on intuitionistic fuzzy graph through some illustrations. Panangiotis Chountas et al. [42] presented the index matrix interpretation of the intuitionistic fuzzy trees. Nagoor Gani et al. [38] presented a method to obtain an intuitionistic fuzzy shortest path length. Borzooei et al. [12] explained the degree of vertices in cartesian product, normal product, tensor product of vague graphs.

2.10. **Bipolar and m-polar Fuzzy graphs.** Hossein Rashmanlou et al. [18] proved that the direct product of two strong bipolar fuzzy graphs is strong and strong product of two complete bipolar fuzzy graph is complete. Sovan Samanta et al. [52] discussed some results of bipolar fuzzy graphs. Ganesh Gorai et al. [16] showed that every product m-polar fuzzy graph is a m-polar fuzzy graph. They also analyzed certain operations like Cartesian product, composition, union, join in m-polar fuzzy graphs [17].

3. Conclusion

In this paper, several results on fuzzy graphs have been referred and this will be a compendium for the researchers to work in the field of fuzzy graph theory.

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Department of Mathematics, Prince Shri Venkateshwara padmavathy Engineering College, Chennai
E-mail address: vmr732002@yahoo.co.in

Mathematics Division, School of Advanced Sciences, VIT University, Chennai
E-mail address: kaliyappan.m@vit.ac.in

Department of Mathematics, Amrita School of Engineering-Coimbatore, Amrita Vishwa Vidyapeetham, Amrita University,India
E-mail address: s_hariharan@ch.amrita.edu