

FREE VIBRATION OF A THERMO-PIEZOELECTRIC PLATE

¹S.Karthikeyan and ^{*2}T.K.Parvathavarthini

¹Department of Mathematics, Government Arts College, Salem - 7,
Tamil Nadu , India.

e-mail : rishabkarthik1@gmail.com

^{*2}Department of Mathematics, Sona College of Technology, Salem-5,
Tamil Nadu ,India.

e-mail : tkparvathavarthini@gmail.com

Abstract: In this paper, free vibration behaviour of thermo piezoelectric plate is analysed using finite element approach. The plate is made up of hexagonal 6mm class. The Constitutive equations for the thermo piezoelectric materials are used to derive finite element equations involving the mechanical, electrical and thermal fields of the structure. This model is developed using four noded elements with five nodal degrees of freedom that is three displacements (u, v, w) with two potentials, electric ϕ and thermal θ coefficients. The solutions for the constitutive equations are assumed in the form of exponential function $e^{i(klx+kmy-pt)}$. Assumed shape functions are used to illustrate the formulation of resulting Eigen value problem. By using thermo piezoelectric material constants, natural frequencies of vibrations are obtained. The numerical results are compared between BaTiO₃ and CdSe for different wave numbers. The damping effect is analysed through the imaginary parts of complex frequencies. Results are tabulated and dispersion curves are drawn. Assumed shape functions are used to illustrate the formulation of resulting Eigen value problem. By using thermo piezoelectric material constants, natural frequencies of vibrations are obtained. The numerical results are compared between BaTiO₃ and CdSe for different wave numbers. The damping effect is analysed through the imaginary parts of complex frequencies. Results are tabulated and dispersion curves are drawn.

AMS subject classification: 74K20, 74G15, 74S05

Key words: Thermo piezoelectric, Finite element method, BaTiO₃/CdSe, complex frequencies, Dispersion curves.

1. Introduction

Thermo piezoelectric crystal plates have been used for engineering structures, particularly in systems involving smart and intelligent structure in the recent years. Due to their ability of converting energy from one form to the other these materials have been widely used in

ultrasonic imaging devices, sensors, actuators, transducers and many other emerging components. These plates have direct application in sensing actuating devices, such as damping and control of vibrations in structures.

Mindlin dealt with high frequency vibrations of Piezo electric crystal plates [1]. The wave propagation in Piezocomposite plate has been studied by H.S. Paul and V.K. Nelson [2]. The vibrations of pyro electric sandwich plate have analysed by V.K. Nelson and S. Karthikeyan [3]. A free vibration of pyro electric layer of hexagonal 6mm class is studied by H.S. Paul and K. Ranganathan [4]. H.S. Paul and G.V. Raman discussed vibrations of pyro electric plate [5]. Free vibrations of piezoelectric layer of hexagonal crystal class 6mm has studied by H.S. Paul, D.P. Raju and T.R. Balakrishnan [6]. Free vibration response of two dimensional cases by Finite element method has been studied by Fernando-Ramirez [7]. Buchannan [8] derived layered versus multi phase magneto-electro-elastic composite plates. Finite element modelling of layered and multi phase axisymmetric temperature distribution case dealt by N. Ganesan, A. Kumaravel and Rajusethuraman [9]. Galerkin finite element derivation for vibration of thermo Piezo electric structures analysed by [10]. Free vibrations of a linear thermo piezoelectric body have studied by J.S. Yang and R.C. Batra [11]. In J.N. Reddy [12] had given a MATLAB algorithm for finite element numerical implementation.

In this paper, the governing equations of linear thermo piezoelectricity motion for free vibration of infinite three dimensional plate is solved by Finite element model using Gaussian quadrature formula and standard concatenation technique with four node elements. Approximations for three displacements with two potentials are constructed for free vibration boundary conditions. The natural frequencies through thickness mode shapes are calculated by solving Eigen value problem for the infinite BaTiO₃/CdSe plate are analysed. Dispersion curves have been drawn for waves propagating at an angle of 30° in x-y plane.

2. Governing equation

The equations of linear thermo piezoelectricity were proposed by Mindlin [1]. The governing equations are,

$$T_{kj,j} = \rho \ddot{u}_k, D_{kl,k} = 0, q_{i,j} + T_0 \ddot{\eta} = 0$$

$$T_{kl} = C_{kjrs} S_{rs} - e_{rkj} E_r - \beta_{ij} \Theta$$

$$D_{i,i} = e_{rkj} S_{rs} + \epsilon_{lk} E_r + p_j \Theta \quad (1)$$

$$\rho \eta = \beta_{ij} S_{rs} + p_j E_r + a \Theta$$

$$\text{Where } a = \rho C_v T_0^{-1}$$

The strain displacement equations are given by,

$$S_1 = \frac{\partial u}{\partial x}, S_2 = \frac{\partial u}{\partial y}, S_3 = \frac{\partial u}{\partial z}, S_4 = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, S_5 = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, S_6 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (2)$$

For crystal class 6mm the material constants of (1) can be written as

$$c_{ij} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \text{ where } c_{66} = \left(\frac{c_{11}-c_{12}}{2}\right), \beta_{ij} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, p_j = \begin{bmatrix} 0 \\ 0 \\ p_3 \end{bmatrix}$$

$$e_{kj} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & e_{15} & 0 \end{bmatrix}, \epsilon_{ij} = \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{11} & 0 \\ 0 & 0 & \epsilon_{11} \end{bmatrix}$$

The electric field vector E_i is related to the electric potential ϕ and the heat flux $q_{i,j}$ as

$$E_1 = -\frac{\partial \phi}{\partial x}, E_2 = -\frac{\partial \phi}{\partial y}, E_3 = -\frac{\partial \phi}{\partial z} \quad (3)$$

$$q_1 = -k_{11} \frac{\partial \Theta}{\partial x}, q_2 = -k_{22} \frac{\partial \Theta}{\partial y}, q_3 = -k_{33} \frac{\partial \Theta}{\partial z} \quad (4)$$

Where the stress and strain tensors are denoted as T_{kj} and S_{kj} , the remaining terms are denoted as u_k the mechanical displacement, D_k the electric displacement, E_k the electric field, T the temperature change from a reference temperature T_0 , $q_{i,j}$ the heat flux, ϕ the electric potential, η the entropy, ρ the mass density and $c_{kijrs}, e_{kj}, \beta_{kj}, \epsilon_{kj}, p_k$ are the elastic, piezoelectric, stress coefficient, dielectric, pyroelectric constants with C_v the specific heat. The usual notation is employed, that a comma followed by a lower case letter indicates spatial derivative and a super imposed dot indicates time derivative.

In the case of free vibration, the absence of body force, free charge densities are assumed. Under these boundary conditions, the equations of motion can be written as,

$$T_{kj,j} = \rho \ddot{u}_k, \quad D_{k,k} = 0, \quad K_{i,j} + T_0 \ddot{\eta} = 0 \quad (5)$$

The solution for an infinite plate wave propagation along $(\ell, m, 0)$ direction are assumed as, by [5]

$$\begin{aligned} u(x, y, z, t) &= U(z) \exp i(klx + kmy - pt), \\ v(x, y, z, t) &= V(z) \exp i(klx + kmy - pt), \\ w(x, y, z, t) &= iW(z) \exp i(klx + kmy - pt), \\ \phi(x, y, z, t) &= i\left(\frac{c_{44}}{e_{33}}\right) \Phi(z) \exp i(klx + kmy - pt), \\ \Theta(x, y, z, t) &= i\left(\frac{c_{44}}{\beta_3}\right) \Theta(z) \exp i(klx + kmy - pt) \end{aligned} \quad (6)$$

Where k is the wave number. p is the angular frequency. $\ell = \cos\theta, m = \sin\theta, \ell^2 + m^2 = 1$ and $i = \sqrt{-1}$. We introduce the non dimensional quantity $\epsilon = kh$ Where h is the thickness of the plate.

3.Finite Element formulation

The derivation of finite element equations and assumed trial solutions in terms of shape functions that corresponds to mechanical displacements, electric potential and thermal are as follows,

$$u_i = [N_u]\{u\}, \phi = [N_\phi]\{\phi\}, \Theta = [N_\theta]\{\Theta\} \quad (7)$$

Where $\{u\}, \{\phi\}$ and $\{\Theta\}$ are unknown nodal point variables while $[N_u], [N_\phi]$ and $[N_\theta]$ are shape functions. In most applications it is convenient to assume all shape functions are same. By the above formulation (6) can be rewritten as,

$$\begin{aligned} u(x, y, z, t) &= [N_u] \exp i(klx + kmy - pt) \{U\}, \\ v(x, y, z, t) &= [N_v] \exp i(klx + kmy - pt) \{V\}, \\ w(x, y, z, t) &= i[N_w] \exp i(klx + kmy - pt) \{W\}, \\ \phi(x, y, z, t) &= i\left(\frac{c_{44}}{e_{33}}\right) [N_\phi] \exp i(klx + kmy - pt) \{\Phi\}, \\ \Theta(x, y, z, t) &= i\left(\frac{c_{44}}{\beta_3}\right) [N_\Theta] \exp i(klx + kmy - pt) \{\Theta\} \end{aligned} \quad (8)$$

In thickness direction, one dimensional Lagrangian interpolation polynomials are used for shape functions, for each of four variables. Power and Fourier series are the most commonly selected functions.

The free vibration quantities vary harmonically with time and a circular frequency. The solution of infinite plate is assumed as,

$$\begin{aligned} u &= z^{j-1} \exp i(klx + kmy - pt) \{U\}, \\ v &= z^{j-1} \exp i(klx + kmy - pt) \{V\}, \\ w &= iz^{j-1} \exp i(klx + kmy - pt) \{W\}, \\ \phi &= i\left(\frac{c_{44}}{e_{33}}\right) z^{j-1} \exp i(klx + kmy - pt) \{\Phi\}, \\ \Theta &= i\left(\frac{c_{44}}{\beta_3}\right) z^{j-1} \exp i(klx + kmy - pt) \{\Theta\}, \text{ for } j = 1 \text{ to } 4 \end{aligned} \quad (9)$$

The equations (2), (3), (4), (5) and (8) are substituted in and integrated over the corresponding volume. We obtain a formulation that corresponds to a completely coupled system could be written in terms of the following Stiffness matrices and Mass matrices are,

$$\begin{bmatrix} M_{uu} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{u} \\ \ddot{\phi} \\ \ddot{\Theta} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_{\Theta u} & -C_{\Theta \phi} & C_{\Theta \Theta} \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{\phi} \\ \dot{\Theta} \end{bmatrix} + \begin{bmatrix} K_{uu} & K_{u\phi} & K_{u\Theta} \\ K_{\phi u} & -K_{\phi\phi} & K_{\phi\Theta} \\ 0 & 0 & K_{\Theta\Theta} \end{bmatrix} \begin{bmatrix} u \\ \phi \\ \Theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (10)$$

$$\text{where } [K_{uu}] = \int_v [B_u]^T [c] [B_u] dV, \quad [K_{u\phi}] = \int_v [B_u]^T [e] [B_\phi] dV,$$

$$[K_{u\Theta}] = \int_v [B_u]^T [\beta] [N_\Theta] dV, \quad [K_{\phi\phi}] = \int_v [B_u]^T [\epsilon] [B_\phi] dV,$$

$$[K_{\phi\Theta}] = \int_v [B_u]^T [p] [N_\Theta]^T dV, \quad [K_{\Theta\Theta}] = \int_v [B_\Theta]^T [k] [B_\Theta] dV,$$

$$[C_{\Theta u}] = \int_v [N_\Theta]^T [\beta]^T [B_u] dV, \quad [C_{\Theta\phi}] = \int_v [N_\Theta]^T [p]^T [B_\phi] dV,$$

$$[C_{\Theta\Theta}] = \int_v [N_\Theta]^T [a] [N_\Theta] dV, \quad [M] = \int_v [N]^T [\rho] [N] dV,$$

$$dV = 2\pi dx dy dz$$

The matrices of equation (9) represent three-dimensional elasticity in Cartesian coordinates are obtained by using the assumed solutions of equation (8). The one-dimensional four node shape function is used and combined with an operator matrix that is based upon the equation (2) to form $[B_u]$ as,

$$[B_u] = [L_u][N_u]$$

$$[B_u] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \end{bmatrix}$$

The matrices $[B_\phi]$ and $[B_\Theta]$ are developed using equation (3) and (4) are as follows,

$$[B_\phi] = [L_\phi][N_\phi] = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}, \quad [B_\Theta] = [L_\Theta][N_\Theta] = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}$$

The matrices $[B_u]$, $[B_\phi]$ and $[B_\Theta]$ are the assumed shape function matrices. The elements of each of these matrices have a very specific form as a result of pre-integrals. The matrices are in fact composed of smaller sub-matrices that consist of the fully suitable four node shape functions, multiplied by various in-plane functions.

The natural frequencies and the corresponding shape functions can be found by solving the above Eigen value problem with no external forces. The explicit form of each of these matrices contained in the stiffness matrix $[K]$ and mass matrix $[M]$.

Assembling the element equations yield the frequency equation is,

$$-p^2[M]\{v\} - ip[C]\{v\} + K\{v\} = 0$$

where

$$[M] = \begin{bmatrix} M_{uu} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, [C] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_{\Theta u} & -C_{\Theta \phi} & C_{\Theta \Theta} \end{bmatrix}, [K] = \begin{bmatrix} K_{uu} & K_{u\phi} & -K_{u\Theta} \\ K_{\phi u} & -K_{\phi\phi} & K_{\phi\Theta} \\ 0 & 0 & K_{\Theta\Theta} \end{bmatrix} \quad (11)$$

by using standard concatenation techniques, the above quadratic problem was solved and resulting eigen values denote the natural frequencies.

4. Analysis and Results

Free vibration natural frequencies of BaTiO₃ are compared with CdSe. The definition for non-dimensional terms is necessary and assumed as follows. The thermo piezoelectric terms are those defined by Ref [2,3 and 4] and the additional non-dimensional terms were derived based upon the governing equations. The non-dimensional terms for the infinite plate analysis are as follows,

$$\overline{C}_{ij} = \frac{c_{ij}}{c_{44}}, K = \frac{\epsilon}{h}, \overline{\beta}_{ij} = \frac{\beta_{ij}}{\beta_3}, \overline{p}_{ij} = \frac{p_{ij}c_{44}}{e_{33}\beta_{33}}, \overline{K}_{ij} = \frac{(\rho c_{44})^{\frac{1}{2}} K_{ij}}{\beta_2^2 h T_0}, \Omega = ph \left(\frac{\rho}{c_{44}} \right)^{\frac{1}{2}}$$

Where Ω is dimensionless frequency. The finite element solution is based upon a four noded element with the displacements u, v, w, ϕ and θ as nodal degrees of freedom and above non dimensional values are solved and the natural non dimensional frequencies are obtained by using MATLAB software with input of dimensionless wave number and the dimensionless frequency is output. The dimensionless frequencies are tabulated and the dispersion curves drawn.

5. Conclusion

This paper consists of the wave propagation of an infinite thermo piezoelectric plate with crystal class 6mm. The governing equations for thermo piezoelectric material have been derived by Finite element formulations. The unknowns (three displacements, electric and thermal) were approximated to give closed form solutions that satisfy the traction free boundary conditions of thermo piezoelectric plate. The stiffness, damping and mass matrices were calculated using Gaussian quadrature formula with standard concatenation techniques, which transforms the yield equations into Eigen value problem. The dimensionless natural frequencies and the corresponding mode shapes have been calculated. The numerical work has been carried out for BaTiO₃ and CdSe. Results are tabulated and Dispersion curves are drawn for dimensionless wave number versus dimensionless frequency. In this study it is observed that the dimensionless frequencies of thermo piezoelectric infinite plate made of BaTiO₃ material is significantly more than that of CdSe material and the damping effect which is analysed

through the imaginary part of complex frequencies shows that the damping effect of the infinite plate made of BaTiO₃ material is lower than that of CdSe material. These compared results in the present study are highly useful to control the unwanted structural vibrations on actuators and sensors.

6. Tables and Figures

The dimensionless frequencies of BaTiO₃ and CdSe materials are in Table

Dimensionless Wave Number ϵ	Dimensionless frequency Ω		
	BaTiO ₃		CdSe
0.05	0.53361128681235 0.00021749300121i	-	0.30253128721012- 0.32563342716902i
0.15	0.53698354905451 0.00237124771210i	-	0.38437625576793- 0.30041618296378i
0.25	0.54373407631354 0.00220693578641i	-	0.42020899019901- 0.32786448233523i
0.5	0.57472841634413 0.00104332697065i	-	0.50662617815921- 0.24494758982220i
0.8	0.62476738105712 0.00127013985056i	-	0.53084995365750- 0.27689748037666i
1	0.66723259669301 0.00325410244402i	-	0.57359847091025- 0.29407254037668i
1.5	0.76792598575410 0.00491490578302i	-	0.62191651576486- 0.23005107944090i
2	0.86118512037020 0.33781488550908i	-	0.64116312942360- 0.26016070264394i
2.7	0.88288842548800 0.33557995034940i	-	0.64037671684552 0.29645526839673i
3	0.89116297732511 0.33506229192796i	-	0.65151110331095 0.31053206910965i
3.5	0.90391504410520 0.33463799593222i	-	0.69857939849850 0.33250186740604i
3.55	0.90513061393953 0.33462092216934i	-	0.69832927393839 0.36460939517817i
4.15	0.91894583573567 0.33473302381784i	-	0.70084880215628 0.35539186156903i

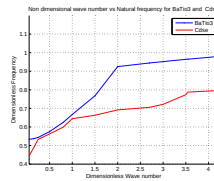


Figure 1:

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