Independent Transversal Restrained Domination in Graphs

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February 10, 2017

Abstract

Let $G$ be a graph. A restrained dominating set intersecting every maximum independent set in $G$ is called an independent transversal restrained dominating set. In this paper, we begin to study this parameter. We calculate $\gamma_{itr}(G)$ for some families of graphs and establish some bounds for $\gamma_{itr}(G)$. Further, the effect of the graph operation edge splitting on $\gamma_{itr}$ is studied.

AMS Subject Classification:Primary 05C69.
Key Words: Restrained dominating set, Independent transversal restrained dominating set, Edge splitting.

1 Introduction

Let $G$ be any graph. For any graph theoretic terminology not defined herein, refer to the book by Murthy [1]. In this paper, we introduce new domination parameter, motivated by independent transversal domination introduced by Ismail Sahul Hamid [3].
A set $D \subseteq V(G)$ is independent if no two vertices of $D$ are adjacent in $G$. The maximum cardinality of an independent set is called the independence number, denoted by $\beta_0(G)$. A set $S \subseteq V(G)$ is called a dominating set of $G$ if $N[S] = V(G)$. The least cardinality of a dominating set is called the domination number, denoted by $\gamma(G)$. A dominating set $S$ is called a restrained dominating set if every vertex in $V - S$ is adjacent to a vertex in $V - S$ and a vertex in $S$. The least cardinality of the restrained dominating set is called the restrained domination number, denoted by $\gamma_r(G)$.

We recall the following theorems required for our study:

**Theorem 1.** [2] Let $G$ be a connected graph of order $n$. Then $\gamma_r(G) = n$ if and only if $G$ is a star.

**Theorem 2.** [2] If $G$ is a graph, then $\gamma_r(G) = 1$ if and only if $G \cong K_1 + H$ where $H$ is a graph with no isolated vertices.

## 2 Independent Transversal Restrained Domination Number

A restrained dominating set $S$ is called an independent transversal restrained dominating set if $S$ intersects every maximum independent set of $G$. The least cardinality of an independent transversal restrained dominating set is called the independent transversal restrained domination number, denoted by $\gamma_{itr}(G)$.

**Example 3.** (i) If $G$ is a complete multipartite graph having $m$ maximum independent sets, then

$$\gamma_{itr}(G) = \begin{cases} 2, & m = 1; \\
            m, & \text{otherwise.} \end{cases}$$

In particular, $\gamma_{itr}(K_n) = n$ and $\gamma_{itr}(K_{m,n}) = 2$, where $m, n \geq 2$.

(ii) For $r \geq 2$, let $G$ be a multi-star of order $n$ having $r$ stars, then $\gamma_{itr}(G) = n - r$. Also $\gamma_{itr}(L(K_{1,n-1})) = n - 1$.

**Theorem 4.** The dominating set $S$ of $G$ is a restrained dominating set of $G$ if and only if $(V - S)$ contains no isolated vertices.

**Theorem 5.** If $n \geq 1$ is an integer, then $\gamma_{itr}(P_n) = n - 2 \left\lfloor \frac{n-1}{3} \right\rfloor$. 


Proof. Let $P_n$ be a path of order $n$. Clearly, every $\beta_0$-set in $P_n$ contains either $v_1$ or $v_2$. Let $S$ be any $\gamma_r$-set of $P_n$, then $v_1, v_n \in S$ and we have two possible cases:

**Case 1:** Suppose $n$ odd. Then $P_n$ contains a unique $\beta_0$-set $D$ containing $v_1$ and so any restrained dominating set $S$ in $P_n$ intersect the $D$ in $v_1$. Therefore, $S$ itself an independent transversal restrained dominating set. Hence, $\gamma_{itr}(P_n) = \gamma_r(P_n) = n - 2\left\lfloor \frac{n-1}{3} \right\rfloor$.

**Case 2:** Suppose $n$ even. Then $P_n$ contains two maximum independent set containing either $v_1$ or $v_n$. In any case, $S$ intersects $\beta_0$-sets at $v_1$ or $v_2$. Therefore, $\gamma_{itr}(P_n) = \gamma_r(P_n) = |S| = n - 2\left\lfloor \frac{n-1}{3} \right\rfloor$. $\square$

**Theorem 6.** If $n \geq 3$, then

$$\gamma_{itr}(C_n) = \begin{cases} 3, & \text{if } n=3; \\ n - 2\left\lfloor \frac{n}{3} \right\rfloor, & \text{otherwise} \end{cases}$$

**Theorem 7.** If $W_n$ is a wheel on $n$ vertices, then

$$\gamma_{itr}(W_n) = \begin{cases} 4, & \text{if } n=4; \\ 3 & \text{otherwise.} \end{cases}$$

Proof. Let $W_n$ be a wheel with $V(W_n) = \{v_1, v_2, \ldots, v_n\}$ and let $v_n$ be the vertex at the center. If $n = 4$, then clearly $\gamma_{itr}(W_4) = 4$. Assume $n \neq 4$. Any $\beta_0$-set in $W_n$ contains exactly one vertex from $\{v_1, v_2\}$. Clearly $W_n$ contains unique maximum independent set containing either $v_1$ or $v_2$. Extending the $\gamma_r$-set $\{v_n\}$ by adding the vertices $v_1, v_2$, we obtain the independent transversal restrained dominating set of minimum cardinality. Thus, $\gamma_{itr}(W_n) = 3$. $\square$

**Theorem 8.** If $G$ is a disconnected graph with components $G_1, G_2, \ldots, G_m$, then $\gamma_{itr}(G) = \min_{1 \leq i \leq m} \{\gamma_{itr}(G_i) + \sum_{j=1, j \neq i}^{m} \gamma_r(G_j)\}$.

Proof. We first assume that $\gamma_{itr}(G_1) + \sum_{j=2}^{m} \gamma(G_j) = \min_{1 \leq i \leq m} \{\gamma_{itr}(G_i) + \sum_{j=1, j \neq i}^{m} \gamma_r(G_j)\}$. Let $S$ be a $\gamma_{itr}$-set of $G_1$ and let $S_j$ be a $\gamma_r$-set of $G_j$, for all $j \geq 2$. Then $S \cup (\cup_{j=2}^{m} S_j)$ is an independent transversal restrained dominating set of $G$ and hence $\gamma_{itr}(G) \leq \gamma_{itr}(G_1) + \sum_{j=2}^{m} \gamma(G_j) = \min_{1 \leq i \leq m} \{\gamma_{itr}(G_i) + \sum_{j=1, j \neq i}^{m} \gamma_r(G_j)\}$.

Conversely, let $S'$ be any independent transversal restrained dominating set of $G$. Then $S'$ must intersect the vertex set $V(G_j)$ of each component $G_j$ of $G$ and $S' \cap V(G_j)$ is a restrained dominating set of $G_j$ for all $j \geq 1$. Further, for at least one $j$, the
set $S' \cap V(G_j)$ must be an independent transversal restrained dominating set of $G_j$, for otherwise each component $G_j$ will have a maximum independent set not intersecting the set $S' \cap V(G_j)$ and so the union of these maximum independent sets from a maximum independent set of $G$ not intersecting $S'$. Hence $|S'| \geq \min_{1 \leq i \leq m} \{\gamma_{itr}(G_j) + \sum_{j=1, j \neq i}^{m} \gamma_r(G_j)\}$. Therefore we obtained that

$$\gamma_{itr}(G) = \min_{1 \leq i \leq r} \{\gamma_{itr}(G_i) + \sum_{j=1, j \neq i}^{r} \gamma(G_j)\}.$$ 

2.1 $\gamma_{itr}$ for some bridged graphs:

**Theorem 9.** If $G$ is a barbell graph of order $2n (n \geq 3)$, then $\gamma_{itr}(G) = n + 1$.

**Proof.** Let $G$ be a barbell graph and let $V(G) = \{u_i, v_i, |1 \leq i \leq n\}$. Then, for $1 \leq i, j \leq n$, the set $S = \{v_i, u_j\}$ will be a maximum independent set in $G$. Let $S$ be an independent transversal restrained dominating set in $G$. Then $S$ must contain all the vertices of one copy of $K_n$. Otherwise, choosing one vertex from each of $K_n$, the set $\{u, v\}$ will be a $\beta_0$-set in $G$ not intersecting $S$, a contradiction. Hence $|S| \geq n + 1$. Conversely, the set $V(G) - \{u_i | 2 \leq i \leq n\}$ is an independent transversal restrained dominating set in $G$ of size $n + 1$. \qed

**Theorem 10.** If $G$ is a firecracker graph of order $nk$, then $\gamma_{itr}(G) = n(k - 1)$.

**Proof.** Let $G$ be a firecracker graph with $nk$ vertices. Clearly each independent set in $G$ contains the leaves of $G$. Let $S$ be an independent transversal restrained dominating set in $G$. Then $V - S$ contains path of order $n$. Hence $|S| \geq n(k - 1)$. Conversely suppose the vertices $v_1, v_2, \ldots, v_n$ denotes the leaves of the the stars which are linked by a bridge. Then $S = V(G) - \{v_1, v_2, \ldots, v_n\}$ is an independent transversal restrained dominating set in $G$ of size $n(k - 1)$. Thus $\gamma_{itr}(G) = n(k - 1)$. \qed

**Corollary 11.** Let $G$ be a centipede graph. Then $\gamma_{itr}(G) = n$.

**Theorem 12.** If $G \cong L_{m,n}$ is a Lollipop graph, then

$$\gamma_{itr}(G) = \begin{cases} 3, & n=2; \\ \left\lceil \frac{n}{3} \right\rceil + 1, & \text{otherwise.} \end{cases}$$
Proof. Let $G$ be a Lollipop graph with $m + n$ vertices. It is easy to note that $\gamma_{itr}(L_{m,2}) = 1$. Suppose $n \neq 2$. we consider the following possible cases:

**Case 1:** Suppose $n \equiv 1 \pmod{3}$. Then $G$ contains an induced path $P_{n+1}$ containing the vertex from the complete graph $K_m$. Further, any $\gamma_{itr}$-set of $P_{n+1}$ will be a an independent transversal restrained dominating set in $G$. Therefore $\gamma_{itr}(G) = \lceil \frac{n}{3} \rceil + 1$.

**Case 2:** Suppose $n \equiv 0 \pmod{3}$. Then $G$ contains an induced path $P_{n+1}$ that contains a leaf from $K_m$. Then the $\gamma_{itr}$-set of $G$ will be same as that of $P_{n+1}$. Hence $\gamma_{itr}(G) = \gamma_{itr}(P_n)$.

**Case 3:** Suppose $n \equiv 2 \pmod{3}$. Then $G$ contains an induced path $P_{n-1}$ that contains no vertex from the complete graph $K_m$. Let $S'$ be a $\gamma_{itr}$-set in $P_{n-1}$. Then $\gamma_{itr}(G) \geq |S'| + 1$. Further the set $S = S' \cup \{v\}$ is an independent transversal restrained dominating set of size $|S'| + 1$. Thus $\gamma_{itr}(G) = \gamma_{itr}(P_{n-1}) + 1 = \lceil \frac{n}{3} \rceil + 1$. \hfill $\square$

### 3 Some Bounds for $\gamma_{itr}(G)$:

**Theorem 13.** For any connected graph $G$, we have $1 \leq \gamma_{itr}(G) \leq n$. Further $\gamma_{itr}(G) = n$ if and only if $G$ is either a complete graph $K_n$ or a star $K_{1,n-1}$.

Proof. Clearly the inequalities are trivial. Assume $\gamma_{itr}(G) = n$. Let $S = \{v\}$ be an independent transversal restrained dominating set in $G$. Then $N(v) = V(G)$ and so $\deg_G(v) = n - 1$. Further $v$ must be in every $\beta_0$-set of $G$. Hence $G = K_1$. Suppose $n \geq 2$. Suppose $\beta_0(G) \geq 2$, then for $v \in V(G)$, consider $S = V - \{v\}$. If $S$ itself an independent transversal restrained dominating set in $G$. Then $\gamma_{itr}(G) \leq n - 1$, a contradiction. Hence $\beta_0(G) = 1$ so that $G \cong K_n$. For otherwise, $S$ should not be a restrained dominating set as it intersects every $\beta_0$-set of $G$. Hence $V(G)$ itself the minimum restrained dominating set of $G$ and so $G \cong K_{1,n-1}$. \hfill $\square$

**Corollary 14.** If $G$ is a non-complete graph which is not a star, then $\gamma_{itr}(G) \leq n - 2$.

**Theorem 15.** If $G$ is a connected graph of order $n \geq 2$, then $2 \leq \gamma_r(G) \leq \gamma_{itr}(G) \leq n$. Also, $\gamma_{itr}(G) = 2$ if and only if $G \cong K_1 + H$, where $H$ is a connected graph with unique $\beta_0$-set.
Theorem 16. Let $T$ be a tree of order $n \geq 3$. Then $\gamma_{itr}(G) = n - 2$ if and only if $T$ is obtained from $P_4$ by adding zero or more leaves to the stems or leaves of the path.

Proof. Let $T$ be a tree of order $n$ such that $\gamma_{itr}(G) = n - 2$. Any $\beta_0$-set in $T$ contains either $v_1$ or $v_n$. Since every independent transversal restrained dominating set in $T$ contains both $v_1$ and $v_n$, it follows that every restrained dominating set in $T$ is an independent transversal restrained dominating set in $T$. If $diam(T) \geq 6$, then $T$ contains an induced path $P_7$. But from Theorem 5, we have $\gamma_{itr}(T) \leq n - 4$, a contradiction.

Further, $T$ cannot be a star because stars are the only trees having diameter 2. Hence we must have $3 \leq diam(T) \leq 5$. Then clearly $T$ contains an induced path $P_4$, say $\langle \{v_1, v_2, v_3, v_4\} \rangle$. Further any other vertex $u$ in $T$ must be adjacent to one of the vertices in $P_4$. For if the vertex $u$ is adjacent to the leaves then $T$ contains an induced path $P_5$ or $P_6$. For otherwise $u$ will be adjacent to $v_2$ or $v_3$. Furthermore, if $diam(T) = 4$ or 5, then $T$ contains an induced path $P_5$ but on adding leaves to the vertices of $P_5$. Hence $T$ is obtained from $P_4$ by adding zero or more leaves to the stems or leaves of the path.

Conversely, it is easy to verify that if $T$ is a tree obtained by adding zero or more leaves to the stems or leaves of the path $P_4$, then $\gamma_{itr}(T) = n - 2$. \qed

Theorem 17. Let $G$ be a connected graph of order $n$ containing a cycle. Then $\gamma_{itr}(G) = n - 2$ if and only if $G$ is $C_4$ or $C_5$ or $G$ can be obtained from $C_3$ by attaching zero or more leaves to at most two of the vertices of the cycle.

Proof. Let $G$ be a cyclic graph such that $\gamma_{itr}(G) = n - 2$. We first observe that any $\beta_0$-set in $G$ contains either $v_1$ or $v_2$. Further, if $G$ contain a cycle $C'$ of order at least 6, then there is an induced path $\langle \{u_1, u_2, \ldots, u_6\} \rangle$ in $G$ such that $V(G) - \{u_1, u_2, u_4, u_5\}$ is an independent transversal restrained dominating set in $G$ of order at least $n - 4$. We now consider the following cases:

Case 1: Suppose $G$ contains either $C_4$ or $C_5$ such that $v_1, v_2, v_3$ and $v_4$ are the consecutive vertices of the cycle. If at least one of the vertex say $v_3$ has a neighbor distinct from $v_2$ and $v_4$, then the set $V(G) - \{v_1, v_2, v_4\}$ is an independent transversal restrained
dominating set in $G$. Hence $\gamma_{itr}(G) \leq n - 1$, a contradiction.

**Case 2**: Suppose $G$ contains a cycle $C_3 = \langle v_1, v_2, v_3 \rangle$. Suppose any vertex in $C_3$, say $v_2$ has a neighbor say $u$ not on the cycle, which is adjacent to another vertex say $w$ not on the cycle. Then $V(G) - \{v_1, u, w\}$ is an independent restrained dominating set in $G$, which is a contradiction. Hence, any vertex of $C_3$ can only be adjacent a pendent vertices not on the cycle. Further, if each vertex of $C_3$ has neighbors not on the cycle, then $V(G) - V(C_3)$ is an independent transversal restrained dominating set in $G$, which is a contradiction. 

\[\square\]

## 4 Edge splitting:

Let $u$ and $v$ be any two vertices in $G$ at a distance 2 apart and let $x$ be a common neighbor of both $u$ and $v$. Then, $\langle \{u, x, v\} \rangle$ is an induced path in $G$. An edge splitting (also called edge lifting) defined on $uxv$ is the process of removing the edges $ux$ and $vx$ while adding the edge $uv$ to $E(G)$. We say that the edges $ux$ and $vx$ are lifted off the vertex $x$. The graph obtained by splitting off the vertex is called as the edge lifted graph, denoted by $G'_{uv}$. Therefore, $V(G'_{uv}) = V(G)$ and $E(G'_{uv}) = (E(G) \setminus \{ux, vx\}) \cup \{uv\}$.

**Example 18.** An edge lift in a graph can cause the independent domination number to increase, decrease, or remain the same. For example, consider the graph $G$ as shown in Fig 1. We note that $\gamma_{itr}(G) = 8$ and that $\gamma_{itr}(G'_{uv}) = 7$ and $\gamma_{itr}(G'_{ab}) = 8$. Further for a cycle $G = C_4$, we have $\gamma_{itr}(G) = 2$ whereas $\gamma_{itr}(G'_{uv}) = 4$.

![Figure 1](https://example.com/figure1.png)

\textbf{Figure 1:}

Let $\mathcal{A}(G) = \{uxv | uxv \text{ is an induced path in } G\}$. By a weak partition of a set, we mean a partition of the set in which some of
the subsets may be empty. To facilitate our study of the effects that edge lifts have on the independent transversal restrained domination number of a graph, we define the following weak partition $\mathcal{A}(G) = (\mathcal{A}^+(G), \mathcal{A}^-(G), \mathcal{A}^0(G))$, according to the effect that the edge lift on each path in $\mathcal{A}(G)$ has on the independent transversal restrained domination number, as follows:

\begin{align*}
\mathcal{A}^+(G) &= \{uxv \in \mathcal{A}(G) | \gamma_{itr}(G_{ux}^w) > \gamma_{itr}(G)\} \\
\mathcal{A}^-(G) &= \{uxv \in \mathcal{A}(G) | \gamma_{itr}(G_{ux}^w) < \gamma_{itr}(G)\} \\
\mathcal{A}^0(G) &= \{uxv \in \mathcal{A}(G) | \gamma_{itr}(G_{ux}^w) = \gamma_{itr}(G)\}.
\end{align*}

**Remark** The set $\mathcal{A}(G) = \emptyset$ if and only if $G$ is a complete graph.

Partitioning the $V(G)$ into subsets of having certain property is one of the direction for the research in graph theory. Further, we can demand each subset in the partition of $V(G)$ to have the property being $\gamma_{itr}$-set and call this partition an independent transversal restrained domatic partition. Further, since the maximum matching of $G$ is always a $\gamma_{itr}$-set of $G$, such partition exists for all graphs except for $K_{2n+1}$ so that asking the maximum order of such partition is reasonable; let us call this maximum order as the independent transversal restrained domatic number and denote it by $d_{itp}(G)$. Now, begin investigating the parameter $d_{itp}$.

**References**


