

Counting Subgroups of a Non-Abelian p-group

$$Z_{p^n} \rtimes Z_p$$

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Abstract

The main goal of this paper is to count subgroups which are isomorphic to cyclic p-group, internal direct product of two cyclic p-group or semi direct product of two cyclic p-group of the non-Abelian p-group $Z_{p^n} \rtimes Z_p$, $n \geq 2$ where p may be even or odd prime, by using simple-theoretical approach.

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1 Introduction

One of the famous problems in combinatorial group theory of 20th century as well as 21th century is to count subgroups of non-abelian finite p-group. In Burnside [10], if p is a prime greater than two, then there are 15 groups of order p^4 up to isomorphism. Five of these are abelian and remaining ten are the non-abelian groups. Out of these

15 groups, only ten are non-abelian groups and one of non-abelian group is given below:

$$Z_{p^3} \rtimes Z_p = \langle a, b : a^{p^3} = b^p = e, bab^{-1} = a^{1+p^2} \rangle$$

Extended form of above said group can be written as $Z_{p^n} \rtimes Z_p = \langle a, b : a^{p^n} = b^p = e, bab^{-1} = a^{1+p^{n-1}} \rangle$ which is a non-abelian p-group of order p^{n+1} for $n \geq 2$ where p may be even or odd prime.

In present work, by using simple number-theoretical approach, the authors deduce formula for the number of subgroups isomorphic to cyclic p-group, internal direct product of two cyclic p-group or semi direct product of two cyclic p-group of the p-group $Z_{p^n} \rtimes Z_p$ $n \geq 2$ where p may be odd or even prime. One can easily find out that $\langle a^{p^{n-1}} \rangle$ and $\langle ba^{ip^{n-1}} \rangle$ where $i = 1, 2, 3, \dots, p$, are cyclic subgroups of order p in group $Z_{p^n} \rtimes Z_p$ except for the group $Z_{2^2} \rtimes Z_2$. Firstly we partition the set S (non-trivial cyclic subgroups of $Z_{p^n} \rtimes Z_p$ except for the group $Z_{2^2} \rtimes Z_2$) into (p+1) partitions.

1.1 Notations and assumptions:-

- (i) $N(H, G)$ is denote the number of subgroups of G isomorphic to H .
- (ii) $Z_{p^n} \rtimes Z_p$, $n \geq 2$ except for the $Z_{2^2} \rtimes Z_2$ is denoted by G .
- (iii) $Z_{p^n} \rtimes Z_p$ is denoted by $\langle a, b : a^{p^n} = b^p = e, bab^{-1} = a^{1+p^{n-1}} \rangle$
- (iv) Rewrite $ba = a^{1+p^{n-1}}b$ as $ab = ba^{(p-1)p^{n-1}+1}$
- (v) $s = (p-1)p^{n-1} + 1$

2 Partition the set S (non-trivial cyclic subgroups of G)

Define two cyclic subgroups H and K in S are equivalent, denoted by $H \sim K$, if and only if $H \cap K$ contains a subgroup of order p (clearly such subgroup is unique and cyclic).

Theorem 1. *The relation $\sim$ between elements of the S is an equivalence relation on S.*

Proof. Obvious. □

Corollary 2. *An equivalence relation $\sim$ on a non-empty set S partitions the set S into the disjoint union of distinct equivalence class.*

Here group G has only $p + 1$ cyclic subgroups of order p , using above theorem we can partition set S into $(p + 1)$ distinct equivalence class and these partition are as follows:

- (a) $\langle a^{p^{n-1}} \rangle = \{H \in S : H \sim \langle a^{p^{n-1}} \rangle\}$ and denoted by class-0.
 (b) $\langle ba^{ip^{n-1}} \rangle = \{H \in S : H \sim \langle ba^{ip^{n-1}} \rangle\}$ and denoted by class- i where $(1 \leq i \leq p)$.

3 Number of cyclic Subgroups of order p^α ($\alpha \geq 1$) in each class

Theorem 3. Prove that there exist unique cyclic subgroup of order p from class-0 can be represent as $\langle a^{p^{n-1}} \rangle$ and cyclic subgroups of order p^α ($\alpha \geq 2$) from class-0 can be represent as $\langle b^j a^{p^{n-\alpha}} \rangle$ where $j = 1, 2, \dots, p$ from G .

Proof. Suppose that $H = \langle b^j a^k \rangle$ be a cyclic subgroup of order p^α in class-0 from G , then $H \sim \langle a^{p^{n-1}} \rangle$. It means $o(b^j a^k) = p^\alpha$ and $(b^j a^k)^{p^{\alpha-1}} \in \langle a^{p^{n-1}} \rangle$. Firstly we find possibilities for j when value of j lies between 1 to p . Here $(b^j a^k)^{p^{\alpha-1}} = b^{jp^{\alpha-1}} a^{k[s(p^{\alpha-1}-1)j + \dots + s^{2j} + s^j + 1]} \in \langle a^{p^{n-1}} \rangle \implies p | jp^{\alpha-1}$. On the basis of j , we have two cases :

Case-1: If $j = p$

Therefore H becomes $H = \langle a^k \rangle$ which cyclic subgroup of order p^α in class-0 from G . Now we find possibilities for k when value of k lies between 1 to p^n . We have $o(a^k) = p^\alpha \implies \frac{p^n}{(p^n, k)} = p^\alpha \implies p^{n-\alpha} = (p^n, k) \implies$ possibilities for k are $p^{\alpha-1}(p-1)$. Each cyclic subgroup of order p^α contains $p^{\alpha-1}(p-1)$ elements of order p^α . Therefore, number of cyclic subgroups in class-0 of order p^α is $p^{\alpha-1}(p-1)/p^{\alpha-1}(p-1) = 1$. This subgroup can be written as $\langle a^{\beta p^{n-\alpha}} \rangle$ where $(p^\alpha, \beta) = 1$, it is easy to check that $\langle a^{\beta p^{n-\alpha}} \rangle = \langle a^{p^{n-\alpha}} \rangle$. Hence, we get cyclic subgroup of order p^α in class-0 from G which is $\langle a^{p^{n-\alpha}} \rangle$.

Case 2: If $(j, p) = 1$

If $\alpha = 1$, then $p \nmid jp^{\alpha-1}$, hence this case is not possible. Hence, number of cyclic subgroups of order p from group in class-0 is one and which is $\langle a^{p^{n-1}} \rangle$. Now only possibility left which is $\alpha > 1$. We have

$$(b^j a^k)^p = b^{jp} a^{k[s(p-1)j + \dots + s^{2j} + s^j + 1]} = a^{k[s(p-1)j + \dots + s^{2j} + s^j + 1]}$$

Subcase 1:- If p is even prime

Here possibility for j is only 1, now we find possibilities for k when value of k lies between 1 to p^n . $(b^j a^k)^2 = a^{k[s^j + 1]}$

Here $s^j = 2^{n-1} + 1$, then $(b^j a^k)^2 = a^{k[2^{n-1}+2]}$.

It is given that $o(b^j a^k) = 2^\alpha$, hence $o((b^j a^k)^2) = \frac{2^\alpha}{(2^\alpha, 2)} = 2^{\alpha-1}$.

$$o(a^{k[2^{n-1}+2]}) = 2^{\alpha-1} \implies \frac{2^\alpha}{(2^n, k[2^{n-1}+2])} = 2^{\alpha-1} \quad (1)$$

It given that $n > 2$ when $p=2$, hence $(2^n, k[2^{n-1}+2]) = (2^n, 2k)$.

From (1) $\frac{2^n}{(2^n, 2k)} = 2^{\alpha-1} \implies 2^{n-\alpha} = (2^{n-1}, k)$.

Hence, the possibilities for k when value of $j=1$ in this case given are $2^{\alpha-1}$.

Therefore, possible element of order 2^α ($\alpha > 1$) from group $Z_{2^n} \rtimes Z_2$ ($n > 2$) which generates a cyclic subgroup of class-0 of order 2^α are $1 \times 2^{\alpha-1} = 2^{\alpha-1}$.

Each cyclic subgroup of order 2^α contains $2^{\alpha-1}(2-1)$ elements of order 2^α . Therefore, number of cyclic subgroups in class-0 of order 2^α is $\frac{2^{\alpha-1}}{2^{\alpha-1}} = 1$. This subgroup can be written as $\langle ba^{\beta 2^{n-\alpha}} \rangle$ where $(2^{\alpha-1}, \beta) = 1$, it is easy to check that $\langle ba^{\beta 2^{n-\alpha}} \rangle = \langle ba^{2^{n-\alpha}} \rangle$.

Hence, we get cyclic subgroup of order 2^α in class-0 from $Z_{2^n} \rtimes Z_2$ which is $\langle a^{2^{n-\alpha}} \rangle$.

Subcase 2:- If p is odd prime

$$(b^j a^k)^p = a^{k[s^{(p-1)j} + \dots + s^{2j} + s^j + 1]},$$

Here $s^j \equiv [p^{n-1}(p-1)j + 1] \pmod{p^n}$

$$(b^j a^k)^p = a^{k[s^{(p-1)j} + \dots + s^{2j} + s^j + 1]} = a^{k[p + p^{n-1} \sum_{j=1}^{p-1} j]}.$$

We have $k[p + p^{n-1}(p-1) \sum_{j=1}^{p-1} j] \equiv kp \pmod{p^n}$.

It is given that $o(b^j a^k) = p^\alpha$, hence $o((b^j a^k)^p) = \frac{p^\alpha}{(p^\alpha, p)} = p^{\alpha-1}$

Hence,

$$o(a^{kp}) = p^{\alpha-1} \implies \frac{p^n}{(p^n, kp)} = p^{\alpha-1} \quad (2)$$

From (2) $\frac{p^n}{(p^n, kp)} = p^{\alpha-1} \implies p^{n-\alpha} = (p^{n-1}, k)$

Hence, possibilities for j are $p-1$ and possibilities for k when value of j is given in this case are $p^{\alpha-1}(p-1)$.

Therefore, possible element of order p^α ($\alpha > 1$) from group $Z_{p^n} \rtimes Z_p$ $n \geq 2$ which generates a cyclic subgroup of class-0 of order p^α are $(p-1) \times p^{\alpha-1}(p-1) = p^{\alpha-1}(p-1)^2$.

Each cyclic subgroup of order p^α contains $p^{\alpha-1}(p-1)$ elements of order p^α . Therefore, number of cyclic subgroups in class-0 of order p^α is $\frac{p^{\alpha-1}(p-1)^2}{p^{\alpha-1}(p-1)} = p-1$.

Hence, number of cyclic subgroups of order p^α ($n \geq \alpha > 1$) from G in class-0 are $1+(p-1)$ when $1 < \alpha \leq n$. Therefore, number of cyclic subgroups of order p^α ($n \geq \alpha > 1$) from G in class-0 are 1 or p according as $\alpha = 1$ or $1 < \alpha \leq n$.

This subgroup can be written as $\langle b^j a^{\beta p^{n-\alpha}} \rangle$ where $(p^{\alpha-1}, \beta) = 1$, it is easy to check that $\langle b^j a^{\beta p^{n-\alpha}} \rangle = \langle b^j a^{p^{n-\alpha}} \rangle$. $\square$

Corollary 3.1. Prove that every cyclic subgroup from class-0 is normal subgroup of G .

Proof. Assume H is cyclic subgroup from class-0.

Case 1:- If H is a cyclic subgroup of order p Therefore, $H = \langle a^{p^{n-1}} \rangle$.

Assume x be an arbitrary element of H and y be an arbitrary element of G , then $x = a^{kp^{n-1}}$ and $y = b^\alpha a^\beta$. So

$$yxy^{-1} = b^\alpha a^\beta a^{kp^{n-1}} a^{-\beta} b^{-\alpha} = b^\alpha a^{kp^{n-1}} b^{-\alpha} = a^{kp^{n-1}(1+p^{n-1})} \in H.$$

Hence H is normal subgroup of G .

Case 2:- Since H is a cyclic subgroup of order p^α ($n \geq \alpha > 1$)

Therefore, $H = \langle b^j a^{p^{n-\alpha}} \rangle$ where $j = 1, 2, 3, \dots, p$.

Assume x be an arbitrary element of H and y be an arbitrary element of G , then $x = (b^j a^{p^{n-\alpha}})^k$ and $y = b^\alpha a^\beta$.

$$yxy^{-1} = b^\alpha a^\beta (b^j a^{p^{n-\alpha}})^k a^{-\beta} b^{-\alpha} = (b^\alpha a^\beta b^j a^{p^{n-\alpha}-\beta} b^{-\alpha})^k = (b^j a^{[p^{n-\alpha}-\beta(s^j-1)]s^{p-\alpha}})^k \quad (3)$$

Here $s^j \equiv [p^{n-1}(p-1)j+1] \pmod{p^n}$ and $s^{p-\alpha} \equiv [p^{n-1}\alpha+1] \pmod{p^n}$

Hence $(p^{n-\alpha} - \beta(s^j - 1))s^{p-\alpha} \equiv (p^{n-\alpha} - \beta p^{n-1}(p-1)j)s^{p-\alpha} \pmod{p^n}$

Subcase 1:- If $\beta = p$, then $(p^{n-\alpha} - \beta(s^j - 1))s^{p-\alpha} \equiv p^{n-\alpha}s^{p-\alpha} \pmod{p^n}$

From (3) becomes $yxy^{-1} = (b^j a^{s^{p-\alpha}p^{n-\alpha}})^k \in \langle b^j a^{s^{p-\alpha}p^{n-\alpha}} \rangle$ where $(s^{p-\alpha}, p) = 1$

Thus we have $yxy^{-1} \in \langle b^j a^{p^{n-\alpha}} \rangle$.

Sub Case 2:- If $(\beta, p) = 1$, then

$$(p^{n-\alpha} - \beta(s^j - 1))s^{p-\alpha} \equiv p^{n-\alpha}(1 - \beta p^{\alpha-1}(p-1)j)s^{p-\alpha} \pmod{p^n}$$

Here $(s^{p-\alpha}, p) = 1$ and $((1 - \beta p^{\alpha-1}(p-1)j), p) = 1$, hence $(1 - \beta p^{\alpha-1}(p-1)j)s^{p-\alpha}, p = 1$, Now equation (3) becomes

$$yxy^{-1} = (b^j a^{[1-\beta p^{\alpha-1}(p-1)j]s^{p-\alpha}})^k \in \langle b^j a^{[1-\beta p^{\alpha-1}(p-1)j]s^{p-\alpha}} \rangle \text{ where } ((1 - \beta p^{\alpha-1}(p-1)j), p) = 1.$$

Thus finally we get $yxy^{-1} \in \langle b^j a^{p^{n-\alpha}} \rangle$. $\square$

Theorem 4. Number of cyclic subgroups of order p^α ($n \geq \alpha \geq 1$) from G in class- i other than class-0 is one or zero according as $\alpha = 1$ or $\alpha > 1$ respectively and $\langle ba^{ip^{n-1}} \rangle$ is the only subgroup in class- i other than class-0.

Proof. Let $H = \langle b^j a^k \rangle$ be a cyclic subgroup of order p^α in class- i other than class-0 from G , then $H \sim \langle ba^{ip^{n-1}} \rangle$. It means $o(b^j a^k) = p^\alpha$ and $(b^j a^k)^{p^{\alpha-1}} \in \langle ba^{ip^{n-1}} \rangle$.

Here $(b^j a^k)^{p^{\alpha-1}} = b^{jp^{\alpha-1}} a^{k[s^{(p^{\alpha-1}-1)j} + \dots + s^j + 1]} \in \langle ba^{ip^{n-1}} \rangle$

$\implies p \nmid jp^{\alpha-1} \implies \alpha = 1$ and $(j, p) = 1$.

There is no cyclic subgroup of order $p^\alpha (n \geq \alpha > 1)$ from $Z_{p^n} \rtimes Z^p (n \geq 2)$ except $Z_{2^2} \rtimes Z_2$ in class- i other than class-0.

Now possibilities left out $\alpha = 1$ and $(j, p) = 1$

Now we can say that $H = \langle b^j a^k \rangle$ with $o(b^j a^k) = p$ and $(j, p) = 1$.

We claim that $p^{n-1} \mid k$

Then $(b^j a^k)^p = b^{jp} a^{k[s^{(p-1)j} + \dots + s^j + 1]} = e \implies p^n \mid k[s^{(p-1)j} + \dots + s^j + 1]$

Here

$$s^j = [(p-1)p^{n-1} + 1]^j \equiv [j(p-1)p^{n-1} + 1] \pmod{p^n} \quad (4)$$

Then $k[s^{(p-1)j} + \dots + s^j + 1] \equiv k[p + p^{\frac{p-1}{2}}(p-1)jp^{n-1}] \equiv kp \pmod{p^n}$

$$p^n \mid kp \implies k = rp^{n-1} \pmod{p^n}, \text{ where } 1 \leq r \leq p \quad (5)$$

By use of $(j, p) = 1$, we have a unique solution of $jx \equiv 1 \pmod{p}$ say j_0 .

$$(b^j a^k)^{j_0} = b^{jj_0} a^{k[s^{(j_0-1)j} + \dots + s^j + 1]} = b^1 a^{k[s^{(j_0-1)j} + \dots + s^j + 1]} \in \langle ba^{ip^{n-1}} \rangle$$

$$\implies k[s^{(j_0-1)j} + \dots + s^j + 1] \equiv ip^{n-1} \pmod{p^n}$$

By using (4) and (5) in the above equation, we get

$$rp^{n-1}[j_0 + jj_0(\frac{j_0-1}{2})(p-1)p^{n-1}] \equiv ip^{n-1} \pmod{p^n}$$

$\implies$

$$rp^{n-1}j_0 \equiv ip^{n-1} \pmod{p^n} \implies rj_0 \equiv i \pmod{p} \implies rj_0j \equiv ij \pmod{p} \implies r \equiv ij \pmod{p} \quad (6)$$

Therefore we have $\langle ba^{ip^{n-1}} \rangle = \langle b^j a^{rp^{n-1}} \rangle$ where $r \equiv ij \pmod{p}$.

Hence we conclude that number of cyclic subgroup of order $p^\alpha (n \geq \alpha \geq 1)$ from G in class- i other than class-0 is one or zero according $\alpha = 1$ or $\alpha > 1$ respectively and $\langle ba^{ip^{n-1}} \rangle$ is the only subgroup in class- i other than class-0. $\square$

4 Main Theorem

Theorem 5. Number of subgroups isomorphic to $Z_{p^\alpha} \times Z_p$ is 1 or 0 according as $1 \leq \alpha < n$ or $\alpha = n$ and isomorphic to $Z_{p^\alpha} \rtimes Z_p$ is 0 or 1 according as $1 \leq \alpha < n$ or $\alpha = n$ from G .

Proof. As we have discussed in Section 3 that non-trivial cyclic subgroup H of order $p^\alpha (n \geq \alpha \geq 1)$ from G becomes normal subgroup if and if this belongs to class-0.

We know that intersection of two non-trivial cyclic subgroups of G is identity if and only both belongs to different class. Hence, we have to choose a non-trivial cyclic subgroup K from class other than class-0 so that product of two subgroups H and K is subgroup because H is normal subgroup of group G . Hence, order K is p . Choice's of H can be classified into three cases:

Case-1: If order of H is p

Total possibilities for H (belong to class-0) is 1 which is $\langle a^{p^{n-1}} \rangle$.

Total possibilities for K (belong to class- i except class-0) are p which are $\langle ba^{ip^{n-1}} \rangle$. Hence HK contains p^2 elements and HK is subgroup.

Now we show that $HK = H \times K$.

$$HK = \{x^j y^k | x = a^{p^{n-1}}, y = ba^{ip^{n-1}}, ba = a^{1+p^{n-1}}b, o(x) = p, o(y) = p\}$$

Now $yx = ba^{ip^{n-1}} a^{p^{n-1}} = ba^{(i+1)p^{n-1}} = a^{p^{n-1}(1+p^{n-1})} ba^{ip^{n-1}} = a^{p^{n-1}} ba^{ip^{n-1}} = xy$. Hence HK becomes

$$\{x^j y^k | xy = yx, o(x) = p, o(y) = p\} = \langle a^{p^{n-1}} \rangle \times \langle ba^{ip^{n-1}} \rangle \cong Z_p \times Z_p.$$

We can easily check that $\langle a^{p^{n-1}} \rangle \times \langle ba^{ip^{n-1}} \rangle = \langle a^{p^{n-1}} \rangle \times \langle ba^{p^{n-1}} \rangle$ when $i \neq 1$.

Therefore, $\langle a^{p^{n-1}} \rangle \times \langle ba^{p^{n-1}} \rangle$ is unique subgroup in which order of H is p and this is isomorphic to $Z_p \times Z_p$.

Case-2: If order of H is $p^\alpha (n > \alpha > 1)$

Total possibilities for H (belong to class-0) is p which is $\langle b^j a^{p^{n-\alpha}} \rangle$ where $j = 1, 2, \dots, p$. Total possibilities for K (belong to class- i except class-0) are p which are $\langle ba^{ip^{n-1}} \rangle$. Hence HK contains $p^{\alpha+1}$ elements and HK is subgroup.

Now we show that $HK = H \times K$.

$$HK = \{x^j y^k | x = b^j a^{p^{n-\alpha}}, y = ba^{ip^{n-1}}, ba = a^{1+p^{n-1}}b, o(x) = p^\alpha, o(y) = p\}.$$

Here $ab = ba^{(p-1)p^{n-1}+1}$ and now assume $s = (p-1)p^{n-1} + 1$.

Hence $s^j = [(p-1)p^{n-1} + 1]^j \equiv [j(p-1)p^{n-1} + 1] \pmod{p^n}$.

Now

$$\begin{aligned} yx &= ba^{ip^{n-1}} b^j a^{p^{n-\alpha}} = b^{1+j} a^{p^{n-1}(is^j+1)} = b^{1+j} a^{p^{n-1}(ij(p-1)p^{n-1}+i+1)} = b^{1+j} a^{p^{n-1}(i+1)} \\ xy &= b^j a^{p^{n-\alpha}} ba^{ip^{n-1}} = b^{j+1} a^{p^{n-1}(s+i)} = b^{j+1} a^{p^{n-1}((p-1)p^{n-1}+1+i)} = b^{1+j} a^{p^{n-1}(i+1)}. \end{aligned}$$

Hence HK becomes

$$\{x^j y^k | xy = yx, o(x) = p^\alpha, o(y) = p\} = \langle b^j a^{p^{n-\alpha}} \rangle \times \langle ba^{ip^{n-1}} \rangle \cong Z_{p^\alpha} \times Z_p.$$

We can easily check that $\langle b^j a^{p^{n-\alpha}} \rangle \times \langle ba^{ip^{n-1}} \rangle = \langle ba^{p^{n-\alpha}} \rangle \times \langle ba^{p^{n-1}} \rangle$ when $i \neq 1 \forall j$.

Therefore, $\langle ba^{p^{n-\alpha}} \rangle \times \langle ba^{p^{n-1}} \rangle$ is unique subgroup in which order of H is p^α ($n > \alpha > 1$) and this is isomorphic to $Z_{p^\alpha} \times Z_p$.

Case 3: If order of H is p^n

Total possibilities for H (belong to class-0) is p which is $\langle b^j a^1 \rangle$ where $j = 1, 2, \dots, p$. Total possibilities for K (belong to class-i except class-0) are p which are $\langle ba^{ip^{n-1}} \rangle$. Hence HK contains $p^{(n+1)}$ elements and HK is subgroup. But Group itself has p^{n+1} elements; hence HK must be group itself. Therefore, $\langle a \rangle \rtimes \langle b \rangle$ is unique subgroup in which order of H is p^n and this is isomorphic to $Z_{p^n} \rtimes Z_p$. $\square$

Theorem 6. Show that for $Z_{p^n} \rtimes Z_p$ except $Z_4 \rtimes Z_2$ where p is any prime as $N(H, Z_{p^n} \rtimes Z_p)$ is

$$\begin{cases} 1 & \text{if } H \cong Z_0 \text{ or } Z_{p^k} \times Z_p \text{ or } Z_{p^n} \rtimes Z_p \text{ where } k = 1, 2, \dots, n-1 \\ p+1 & \text{if } H \cong Z_p \\ p & \text{if } H \cong Z_{p^k} \text{ where } k = 2, 3, \dots, n. \end{cases}$$

Proof. By use of theorem 3, 4 and 5, we get required result. $\square$

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