

Δ^* Homeomorphism in Topological Spaces

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February 10, 2017

Abstract

The objective of this article is to introduce and explore the idea of a new class of homeomorphisms called, Δ^* -homeomorphisms using Δ^* -open maps via Δ^* -closed sets in topological spaces. The association of Δ^* -homeomorphisms with other existing maps and the applications of Δ^* -homeomorphism are also analysed in this paper.

AMS Subject Classification: 54A05, 54C05, 54C10

Key Words and Phrases : Δ^* -closed set, Δ^* -continuous, Δ^* -open map, Δ^* -irresolute map and Δ^* T_δ -space

1 Introduction

The conception of generalised closed maps in topological spaces was introduced by S.R.Malghan [4] in 1982. The idea of generalised homeomorphism and gc-homeomorphism was introduced and

studied by H.Maki et.al., [5] in 1991. Further the concept of δ -homeomorphism, $g\delta$ -homeomorphism, $\alpha\hat{g}$ -homeomorphism and $\delta\hat{g}$ -homeomorphism maps were published in [2], [1] and [3]. In this article, a new class of homeomorphisms called, Δ^* -homeomorphisms are established by introducing a new class of open maps namely, Δ^* -open maps. The association of Δ^* -homeomorphism with various existing homeomorphisms are offered in this paper. Moreover some applications of this newly defined Δ^* -homeomorphism are described in this paper. The topological spaces stated in this paper have no separation axioms if it is not mentioned.

Remark : The authors initially denoted Δ^* -closed sets by $\delta(\delta g)^*$ -closed sets [6].

2 Preliminaries

Definition 2.1 Let A be a subset of a topological space (X, τ) . Then it is called a Δ^* -closed set [6] if $\delta\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ where U is δg -open in (X, τ) . The class of all Δ^* -closed sets of (X, τ) is denoted by $\Delta^*\mathcal{C}(X, \tau)$. The complement of Δ^* -closed sets are called Δ^* -open sets.

Definition 2.2 A space (X, τ) is said to be a Δ^*T_δ -space [7] if every Δ^* -closed subset of (X, τ) is δ -closed in (X, τ) .

Definition 2.3 A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be Δ^* -continuous [8] if the inverse image of every closed set in (Y, σ) is Δ^* -closed in (X, τ) .

Definition 2.4 A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is called a Δ^* -irresolute map [9] if $f(V)$ is Δ^* -open set in (Y, σ) for every Δ^* -open set V in (X, τ) .

Definition 2.5 A bijection map $f : (X, \tau) \rightarrow (Y, \sigma)$ is called a

- i) Δ^* -homeomorphism [10] if f is bijection, Δ^* -open and Δ^* -continuous.
- ii) $g\delta$ -homeomorphism [2] if f is bijection, $g\delta$ -open and $g\delta$ -continuous.
- iii) $\delta\hat{g}$ -homeomorphism [3] if f is bijection, $\delta\hat{g}$ -open and $\delta\hat{g}$ -continuous.

continuous.

iv) $\alpha\hat{g}$ -homeomorphism [1] if f is bijection, $\alpha\hat{g}$ -open and $\alpha\hat{g}$ -continuous.

v) δg^* -homeomorphism [11] if f is bijection, δg^* -open and δg^* -continuous.

vi) gc -homeomorphism [5] if both f and f^{-1} are g -irresolute.

3 Δ^* -Homeomorphism

Definition 3.1 A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be a Δ^* -open map if $f(V)$ is a Δ^* -open set in (Y, σ) for every open set V in (X, τ) .

Definition 3.2 A bijection map $f : (X, \tau) \rightarrow (Y, \sigma)$ is called a Δ^* -homeomorphism if f is both Δ^* -continuous and Δ^* -open map.

Proposition 3.3 Every Δ^* -homeomorphism is a $g\delta$ -homeomorphism but not conversely.

Proof : Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a Δ^* -homeomorphism. Then f is bijective, Δ^* -continuous and Δ^* -open map. Let V be a closed set in (Y, σ) . Then $f^{-1}(V)$ is Δ^* -closed in (X, τ) . Since every Δ^* -closed set is $g\delta$ -closed (Proposition 3.6 of [6]), $f^{-1}(V)$ is $g\delta$ -closed in (X, τ) . This implies that f is $g\delta$ -continuous. Let U be an open set in (X, τ) . Then $f(U)$ is $g\delta$ -open set and hence it is $g\delta$ -open in (Y, σ) . Thus f is a $g\delta$ -open map. Hence f is a $g\delta$ -homeomorphism.

Counter Example 3.4 Let $X = \{p, q, r\} = Y$ with $\tau = \{\phi, X, \{p\}\}$, and $\sigma = \{\phi, Y, \{p\}, \{p, q\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then f is $g\delta$ -homeomorphism but not Δ^* -homeomorphism since for the closed set $\{r\}$ in (Y, σ) , $f^{-1}\{r\} = \{r\}$ is not Δ^* -closed in (X, τ) .

Proposition 3.5 i) Every δ -homeomorphism is a Δ^* -homeomorphism.
ii) Every δg^* -homeomorphism is a Δ^* -homeomorphism.

Proof : i) Follows from the fact that every δ -continuous is Δ^* -continuous (Proposition 3.3 of [8]) and every δ -open is Δ^* -open (Proposition 3.2 of [6]). ii) Follows from the fact that every δg^* -continuous is Δ^* -continuous (Proposition 3.5 of [8]) and every δg^* -open is Δ^* -open (Proposition 3.4 of [6]).

Remark 3.6 The converse of the above proposition is not true which is shown by the following example.

Counter Example 3.7 Let $X = \{p, q, r\} = Y$ with $\tau = \{\phi, X, \{p\}, \{p, q\}, \{p, r\}\}$ and $\sigma = \{\phi, Y, \{q\}, \{r\}\}$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then f is a Δ^* -homeomorphism but not a δ -homeomorphism as f is not δ -continuous, since for the closed set q in (Y, σ) , $f^{-1}\{q\} = \{q\}$ is not δ -closed in (X, τ) . Also f is a Δ^* -homeomorphism but not a δg^* -homeomorphism as f is not δg -continuous, since for the closed set $\{q\}$ in (Y, σ) , $f^{-1}\{q\} = \{q\}$ is not δg^* -closed in (X, τ) .

Remark 3.8 The Δ^* -homeomorphism map and $\delta\hat{g}$ -homeomorphism map are independent which is proved by the example given below.

Counter Example 3.9 Let $X = \{p, q, r\} = Y$ with $\tau = \{\phi, X, \{p\}\}$ and $\sigma = \{\phi, Y, \{p\}, \{q\}, \{p, q\}\}$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then f is a $\delta\hat{g}$ -homeomorphism but not a Δ^* -homeomorphism as f is not Δ^* -continuous, since for the closed set $\{r\}$ in (Y, σ) , $f^{-1}\{r\} = \{r\}$ is not Δ^* -closed in (X, τ) .

Counter Example 3.10 Let $X = \{p, q, r\} = Y$ with $\tau = \{\phi, X, \{p\}, \{p, q\}, \{p, r\}\}$ and $\sigma = \{\phi, Y, \{p\}, \{p, q\}, \{p, r\}\}$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then f is a Δ^* -homeomorphism but not a $\delta\hat{g}$ -homeomorphism map since for the closed set $\{q\}$ in (Y, σ) , $f^{-1}\{q\} = \{q\}$ is not $\delta\hat{g}$ -closed in (X, τ) .

Remark 3.11 The Δ^* -homeomorphism map and $\alpha\hat{g}$ -homeomorphism map are independent which is proved by the following examples.

Counter Example 3.12 Let $X = \{p, q, r\} = Y$ with $\tau = \{\phi, X, \{p\}\}$ and $\sigma = \{\phi, Y, \{p\}, \{q\}, \{p, q\}\}$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then f is a $\alpha\hat{g}$ -homeomorphism but not a Δ^* -homeomorphism as f is not Δ^* -continuous, since for the closed set $\{r\}$ in (Y, σ) , $f^{-1}\{r\} = \{r\}$ is not Δ^* -closed in (X, τ) .

Counter Example 3.13 Let $X = \{p, q, r\} = Y$ with $\tau = \{\phi, X, \{p\}, \{q\}, \{p, q\}, \{p, r\}\}$ and $\sigma = \{\phi, Y, \{p\}, \{p, q\}, \{p, r\}\}$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then f is a Δ^* -homeomorphism but not a $\alpha\hat{g}$ -homeomorphism as f is not a $\alpha\hat{g}$

-open map, since the image of the open set $\{q\}$ in (X, τ) is not $\alpha\hat{g}$ -open in (Y, σ) .

Remark 3.14 The Δ^* -homeomorphism and homeomorphism are independent as proved by the following examples.

Counter Example 3.15 Let $X = \{p, q, r\} = Y$ with $\tau = \{\emptyset, X, \{p\}, \{p, q\}$ and $\sigma = \{\emptyset, Y, \{p\}, \{q\}, \{p, q\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then f is a Δ^* -homeomorphism but not a homeomorphism as f is not continuous, since for the closed set $\{p, r\}$ in (Y, σ) , $f^{-1}\{p, r\} = \{p, r\}$ is not closed in (X, τ) .

Counter Example 3.16 Let $X = \{p, q, r, s\} = Y$ with $\tau = \{\emptyset, X, \{p\}, \{q\}, \{p, q\}, \{p, r\}, \{p, q, r\}, \{p, r, s\}$ and $\sigma = \{\emptyset, Y, \{p\}, \{r\}, \{p, q\}, \{p, r\}, \{p, q, r\}, \{p, r, s\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then f is a homeomorphism but not a Δ^* -homeomorphism as f is not a Δ^* -open map, since the image of the open set $\{p, r, s\}$ in (X, τ) , $f(\{p, r, s\}) = \{p, r, s\}$ is not Δ^* -open in (Y, σ) .

4 Applications of Δ^* -Homeomorphism

Definition 4.1 A bijection map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be a $\Delta^*\mathcal{C}$ -homeomorphism if both f and f^{-1} are Δ^* -irresolute maps. The family of all $\Delta^*\mathcal{C}$ -homeomorphism of a topological space (X, τ) onto itself are denoted by $\Delta^*\mathcal{C}\mathcal{H}(X, \tau)$.

Theorem 4.2 Every Δ^* -homeomorphism from a $_{\Delta^*}T_\delta$ -space into another $_{\Delta^*}T_\delta$ -space is a homeomorphism.

Proof : Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a Δ^* -homeomorphism. Then f is bijective, Δ^* -open and Δ^* -continuous. Let U be an open set in (X, τ) . Since f is Δ^* -open and (Y, σ) is a $_{\Delta^*}T_\delta$ -space, $f(U)$ is δ -open which implies that $f(U)$ is open in (Y, σ) . Therefore f is an open map. Let V be a closed set in (Y, σ) . Since f is Δ^* -continuous and (X, τ) is a $_{\Delta^*}T_\delta$ -space, $f^{-1}\{V\}$ is δ -closed in (X, τ) which implies that $f^{-1}\{V\}$ is closed in (X, τ) . Therefore f is continuous. Hence f is a homeomorphism.

Theorem 4.3 If $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ are Δ^* -homeomorphisms where (Y, σ) is a $_{\Delta^*}T_\delta$ -space then the composition map $(g \circ f) : (X, \tau) \rightarrow (Z, \eta)$ is a Δ^* -homeomorphism.

Proof : Let U be any open set in (X, τ) . Since f is Δ^* -open, $f(U)$ is a Δ^* -open set in (Y, σ) . Since (Y, σ) is a $_{\Delta^*}T_\delta$ -space, $f(U)$ is δ -open in (Y, σ) . Since every δ -open is open, $f(U)$ is open in (Y, σ) . Also $g[f(U)]$ is Δ^* -open in (Z, η) , since g is Δ^* -open. Hence $(g \circ f)$ is a Δ^* -open map. Let V be a closed set in (Z, η) . Since g is Δ^* -continuous, $f^{-1}\{V\}$ is Δ^* -closed in (Y, σ) . Since (Y, σ) is a $_{\Delta^*}T_\delta$ -space, $f^{-1}\{V\}$ is Δ^* -closed and hence it is closed in (Y, σ) . Since f is Δ^* -continuous, $f^{-1}(g^{-1}\{V\}) = (g \circ f)^{-1}(V)$ is Δ^* -closed in (X, τ) . Therefore $(g \circ f)$ is Δ^* -continuous. Hence $(g \circ f)$ is a Δ^* -homeomorphism.

Remark 4.4 The $\Delta^*\mathcal{C}$ -homeomorphism and $\delta g^*\mathcal{C}$ -homeomorphism are independent which is proved by the following examples.

Counter example 4.5 Let $X = \{p, q, r\} = Y$ with $\tau = \{\phi, X, \{p\}\}$ and $\sigma = \{\phi, Y, \{p\}, \{p, q\}, \{p, r\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then f is a $\delta g^*\mathcal{C}$ -homeomorphism but not a $\Delta^*\mathcal{C}$ -homeomorphism as f is not a Δ^* -irresolute since for the Δ^* -open set $\{p, r\}$ in (Y, σ) , $f^{-1}\{p, r\} = \{p, r\}$ is not Δ^* -open in (X, τ) .

Counter example 4.6 Let $X = \{p, q, r, s\} = Y$ with $\tau = \{\phi, X, \{p\}, \{p, q\}\}$ and $\sigma = \{\phi, Y, \{p, q, r\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then f is a $\Delta^*\mathcal{C}$ -homeomorphism but not a $\delta g^*\mathcal{C}$ -homeomorphism as f is not a δg^* -irresolute since for the δg^* -closed set $\{d\}$ in (Y, σ) , $f^{-1}(\{d\}) = \{d\}$ is not δg^* -closed in (X, τ) .

Proposition 4.7 The composition of two $\Delta^*\mathcal{C}$ -homeomorphisms is $\Delta^*\mathcal{C}$ -homeomorphism.

Proof : Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ be $\Delta^*\mathcal{C}$ -homeomorphisms. Let V be a Δ^* -open set in (Z, η) . Since g is Δ^* -irresolute, $g^{-1}(\{V\})$ is Δ^* -open in (Y, σ) . Since f is a Δ^* -irresolute and bijective map, $f^{-1}(g^{-1}\{V\})$ is Δ^* -open in (X, τ) . That is $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}\{V\})$ is Δ^* -open in (X, τ) . This implies that $(g \circ f)$ is a Δ^* -irresolute. Let G be a Δ^* -open set in (X, τ) . Since f^{-1} is Δ^* -irresolute, is $(f^{-1})^{-1}(G)$ is Δ^* -open in (Y, σ) . That is $f(G)$ is Δ^* -open in (Y, σ) . Since g^{-1} is Δ^* -irresolute,

$(g^{-1})^{-1}(f(G))$ is Δ^* -open in (Z, η) . That is $g(f(G))$ is Δ^* -open in (Z, η) . Therefore $(g \circ f)(G)$ is Δ^* -open in (Z, η) . This implies that $((g \circ f)^{-1})^{-1}(G)$ is Δ^* -open in (Z, η) . Thus the mapping $(g \circ f)^{-1} : (Z, \eta) \rightarrow (X, \tau)$ is Δ^* -irresolute. Hence the composition map $(g \circ f)$ is a $\Delta^*\mathcal{C}$ -homeomorphism.

Theorem 4.8 The set $\Delta^*\mathcal{C}\text{-}\mathcal{H}(X, \tau)$ is a group under the composition of maps.

Proof : By defining a binary operation $*$: $\Delta^*\mathcal{C}\text{-}\mathcal{H}(X, \tau) \times \Delta^*\mathcal{C}\text{-}\mathcal{H}(X, \tau) \rightarrow \Delta^*\mathcal{C}\text{-}\mathcal{H}(X, \tau)$ by $(f * g) = (g \circ f)$ for every $f, g \in \Delta^*\mathcal{C}\text{-}\mathcal{H}(X, \tau)$ where $\circ$ is the usual operation of composition of maps and by using the Proposition 4.7, it can be observed that $(g \circ f) \in \Delta^*\mathcal{C}\text{-}\mathcal{H}(X, \tau)$. We know that the composition of maps is associative and the identity map $I : (X, \tau) \rightarrow (X, \tau)$ is the identity element belongs to $\Delta^*\mathcal{C}\text{-}\mathcal{H}(X, \tau)$. If $f \in \Delta^*\mathcal{C}\text{-}\mathcal{H}(X, \tau)$ then $f^{-1} \in \Delta^*\mathcal{C}\text{-}\mathcal{H}(X, \tau)$ such that $(f \circ f^{-1}) = (f^{-1} \circ f) = I$. Therefore the inverse exists for each element of $\Delta^*\mathcal{C}\text{-}\mathcal{H}(X, \tau)$. Hence $\Delta^*\mathcal{C}\text{-}\mathcal{H}(X, \tau)$ is a group under the operation of composition of maps.

Theorem 4.9 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a $\Delta^*\mathcal{C}$ -homeomorphism. Then f becomes an isomorphism from the group $\Delta^*\mathcal{C}\text{-}\mathcal{H}(X, \tau)$ onto the group $\Delta^*\mathcal{C}\text{-}\mathcal{H}(Y, \sigma)$.

Proof : Using the map f , we define a map $\theta_f : \Delta^*\mathcal{C}\text{-}\mathcal{H}(X, \tau) \rightarrow \Delta^*\mathcal{C}\text{-}\mathcal{H}(Y, \sigma)$ by $\theta_f(h) = (f \circ h \circ f^{-1})$ for every $h \in \Delta^*\mathcal{C}\text{-}\mathcal{H}(X, \tau)$. Then θ_f is a bijective map. Further for every $h_1, h_2 \in \Delta^*\mathcal{C}\text{-}\mathcal{H}(X, \tau)$, $\theta_f(h_1, h_2) = f \circ (h_1 \circ h_2) \circ f^{-1} = (f \circ h_1 \circ f^{-1}) \circ (f \circ h_2 \circ f^{-1}) = \theta_f(h_1) \circ \theta_f(h_2)$. Therefore θ_f is a homomorphism. Hence θ_f is an isomorphism created by f .

5 Conclusion

As an application of Δ^* -closed sets it is emphasized that the composition of two Δ^* -homeomorphisms is again a Δ^* -homeomorphism using Δ^*T_δ -space. In addition, it is shown that the family of all $\Delta^*\mathcal{C}$ -homeomorphisms of a topological space (X, τ) onto itself denoted by $\Delta^*\mathcal{C}\text{-}\mathcal{H}(X, \tau)$ forms a group under composition of maps.

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