

Oscillation and Property B for Third Order Difference Equations with Advanced Arguments

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Abstract

In this paper we present sufficient conditions for the third order nonlinear advanced difference equation have property B or to be oscillatory. The results obtained are improve known results for delay difference equations and for ordinary difference equations. Examples are provided to illustrate the main results.

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1 Introduction

Consider the nonlinear third order difference equation

$$\Delta(a_n(\Delta(b_n\Delta x_n))^\alpha) - p_nf(x_{\sigma(n)}) = 0, \quad n \geq n_0, \quad (1)$$

where n_0 is a nonnegative integer, where α is quotient of odd positive integers, $\{a_n\}$, $\{b_n\}$ and $\{p_n\}$ are positive real sequences, and

$\{\sigma(n)\}$ is a nondecreasing sequence of integers with $\sigma(n) \geq n$ for all $n \geq n_0$, f is a real valued continuous nondecreasing function, $uf(u) > 0$ for $u \neq 0$, and $f(uv) \geq f(u)f(v)$ for $uv > 0$, and $\sum_{n=n_0}^{\infty} \frac{1}{a_n^{1/\alpha}} = \sum_{n=n_0}^{\infty} \frac{1}{b_n} = \infty$.

By a solution of equation (1), we mean a real sequence $\{x_n\}$ which is defined for all $n \geq n_0$ and satisfies equation (1). A non-trivial solution of equation (1) is called nonoscillatory if it is either eventually positive nor eventually negative, and otherwise it is called oscillatory.

The study of oscillatory properties of third order and higher order linear difference equations received less attention, see for example [1, 2] and the references cited therein. The investigation of property B for the linear third order difference equation

$$\Delta^3 x_n - p_n y_{\sigma(n)} = 0 \quad (2)$$

with $\sigma(n) = n + 1$ discussed in [7]. By property B of equation (2) it is meant that every positive solution $\{x_n\}$ of equation (2) is strongly increasing, that is $\Delta x_n > 0$, $\Delta^2 x_n \geq 0$, $\Delta^3 x_n \geq 0$.

In [3–6], the authors presented sharp criteria for the qualitative properties of solutions of third order difference equations with delay argument. In this paper, we extend the technique that yields property B of equation (2) to equation (1). Thus the results essentially improve and complement many known results for both delay and ordinary difference equations.

2 Preliminary Results

In this section, first we classify the possible nonoscillatory solutions of equation (1) which follows from the discrete Kneser's theorem [1]. We introduce the following classes of nonoscillatory (let us say positive) solutions of equation (1):

$$x_n \in S_1 \iff \Delta x_n > 0, \Delta(b_n \Delta x_n) < 0, \Delta(a_n \Delta(b_n \Delta x_n)) > 0,$$

and

$$x_n \in S_3 \iff \Delta x_n > 0, \Delta(b_n \Delta x_n) > 0, \Delta(a_n \Delta(b_n \Delta x_n)) > 0,$$

for all $n \geq N \geq n_0$.

Lemma 1. Assume that $\{x_n\}$ is an eventually positive solution of equation (1), then $\{x_n\} \in S_1$ or $\{x_n\} \in S_3$.

In the following we derive some important monotonic properties and estimates of nonoscillatory solutions, that will be used in the main results. Denote

$$A_n = \sum_{s=N}^{n-1} \frac{1}{a_s^{1/\alpha}}, \quad B_n = \sum_{s=N}^{n-1} \frac{1}{b_s}, \quad C_n = \sum_{s=N}^{n-1} \frac{1}{b_s} \sum_{t=N}^{s-1} \frac{1}{a_t^{1/\alpha}}$$

and $P_n = \frac{1}{a_n^{1/\alpha}} \left(\sum_{s=n}^{\infty} p_s \right)^{\frac{1}{\alpha}}$, for N is large enough.

Lemma 2. Let $\{x_n\} \in S_3$ be a positive solution of equation (1), and

$$\sum_{n=N}^{\infty} p_n f(C_{\sigma(n)}) = \infty. \quad (3)$$

Then $\{\frac{x_n}{C_n}\}$ is eventually increasing for all $n \geq N$.

Lemma 3. If $\{x_n\} \in S_1$ be a positive solution of equation (1), then $\{\frac{x_n}{B_n}\}$ is eventually decreasing.

The proof of last two lemmas are similar to that of in [6].

3 Criteria for Property B

In this section, we provide several criteria for the class S_1 of equation (1) to be empty.

Theorem 4. Assume that $\lim_{u \rightarrow \pm\infty} \frac{u}{f^{1/\alpha}(u)} = k_1 < \infty$, and

$$\sum_{n=N}^{\infty} \frac{1}{b_n} \sum_{s=n}^{\infty} \frac{1}{a_s^{1/\alpha}} \left[\sum_{t=s}^{\infty} p_t \right]^{1/\alpha} = \infty, \quad (4)$$

If

$$\limsup_{n \rightarrow \infty} \left\{ f^{1/\alpha} \left(\frac{1}{B_{\sigma(n)}} \right) \sum_{s=N}^{n-1} f^{1/\alpha}(B_{\sigma(s)}) B_{s+1} P_s + \sum_{s=n}^{\sigma(n)-1} B_{s+1} P_s + B_{\sigma(n)} \sum_{n=\sigma(n)}^{\infty} P_n \right\} > k_1$$

then equation (1) has property B.

Proof. Assume equation (1) has a positive solution $\{x_n\} \in S_1$ for all $n \geq N$. Summation of equation (1) from n to ∞ twice yields

$$b_n \Delta x_n \geq \sum_{n=N}^{\infty} \frac{1}{a_n^{1/\alpha}} \left[\sum_{s=n}^{\infty} p_s f(x_{\sigma(s)}) \right]^{1/\alpha} = \sum_{n=N}^{\infty} f^{1/\alpha}(x_{\sigma(n)}) P_n,$$

where we have used the monotonicity of f . Summing again from N to $n-1$, and then changing the order of summation we have

$$x_n \geq \sum_{s=N}^{n-1} f^{1/\alpha}(x_{\sigma(s)}) P_s B_{s+1} + B_n \sum_{s=n}^{\infty} f^{1/\alpha}(x_{\sigma(s)}) P_s.$$

Using the monotonicities of $\{x_n\}$ and $\{\frac{x_n}{B_n}\}$, we obtain

$$\begin{aligned} \frac{x_{\sigma(n)}}{f^{1/\alpha}(x_{\sigma(n)})} &\geq f^{1/\alpha} \left(\frac{1}{B_{\sigma(n)}} \right) \sum_{s=N}^{n-1} f^{1/\alpha}(B_{\sigma(s)}) P_s B_{s+1} \\ &\quad + \sum_{s=N}^{\sigma(n)-1} P_s B_{s+1} + B_{\sigma(n)} \sum_{s=\sigma(n)}^{\infty} P_s. \end{aligned} \quad (5)$$

From the assumption (4), we see that $x_n \rightarrow \infty$ as $n \rightarrow \infty$. Taking $\limsup$ as $n \rightarrow \infty$ on both sides of the inequality (5), we get a contradiction and the proof is now complete. $\square$

Theorem 5. Assume that $\lim_{u \rightarrow 0} \frac{u}{f^{1/\alpha}(u)} = k_2 < \infty$, and

$$\sum_{n=N}^{\infty} \frac{1}{a_n^{1/\alpha}} \left[\sum_{s=n}^{\infty} p_s f(B_{\sigma(s)}) \right]^{1/\alpha} = \infty, \quad (6)$$

$$\begin{aligned} \text{If } \limsup_{n \rightarrow \infty} \left\{ \frac{1}{B_{\sigma(n)}} \sum_{s=N}^{n-1} f^{1/\alpha}(B_{\sigma(s)}) B_{s+1} P_s \right. \\ \left. + \frac{f^{1/\alpha}(B_{\sigma(n)})}{B_{\sigma(n)}} \sum_{s=n}^{\sigma(n)-1} B_{s+1} P_s + f^{1/\alpha}(B_{\sigma(n)}) \sum_{n=\sigma(n)}^{\infty} P_n \right\} > k_2, \end{aligned}$$

then equation (1) has property B.

Proof. Assume that $\{x_n\}$ is an eventually positive solution of equation (1) such that $\{x_n\} \in S_1$ for $n \geq N$. By Lemma 3, the sequence $\{\frac{x_n}{B_n}\}$ is decreasing and we shall show that (6) implies

$$\lim_{n \rightarrow \infty} \frac{x_n}{B_n} = 0. \quad (7)$$

If not, then $\lim_{n \rightarrow \infty} \frac{x_n}{B_n} = d > 0$. Then $\frac{x_n}{B_n} \geq d$ and therefore

$$f(x_{\sigma(n)}) = f\left(\frac{x_{\sigma(n)}}{B_{\sigma(n)}} B_{\sigma(n)}\right) \geq f(d)f(B_{\sigma(n)}).$$

Further, summing equation (1) twice yields

$$b_N \Delta x_N \geq \sum_{n=N}^{\infty} \frac{1}{a_n^{1/\alpha}} \left[\sum_{s=n}^{\infty} p_s f(x_{\sigma(s)}) \right]^{1/\alpha} \geq f(d) \sum_{n=N}^{\infty} \frac{1}{a_n^{1/\alpha}} \left[\sum_{s=n}^{\infty} p_s f(B_{\sigma(s)}) \right]^{1/\alpha},$$

which contradicts (6), and we conclude that (7) holds. Set $z_n = \frac{x_{\sigma(n)}}{B_{\sigma(n)}}$, condition (5) implies

$$\begin{aligned} \frac{z_n}{f^{1/\alpha}(z_n)} &\geq \frac{1}{B_{\sigma(n)}} \sum_{s=N}^{n-1} f^{1/\alpha}(B_{\sigma(s)}) P_s B_{s+1} + \frac{f^{1/\alpha}(B_{\sigma(n)})}{B_{\sigma(n)}} \sum_{s=N}^{\sigma(n)-1} P_s B_{s+1} \\ &\quad + f^{1/\alpha}(B_{\sigma(n)}) \sum_{s=\sigma(n)}^{\infty} P_s. \end{aligned} \quad (8)$$

Taking $\limsup$ as $n \rightarrow \infty$ in (8), we have a contradiction with the assumptions of the theorem. The proof is now complete. $\square$

Corollary 6. *Let condition (4) holds and*

$$\limsup_{n \rightarrow \infty} \left\{ B_{\sigma(n)}^{-\beta/\alpha} \sum_{s=N}^{n-1} B_{\sigma(s)}^{\beta/\alpha} B_{s+1} P_s + \sum_{s=n}^{\sigma(n)-1} B_{s+1} P_s + B_{\sigma(n)} \sum_{n=\sigma(n)}^{\infty} P_n \right\} > 0$$

then the superlinear difference equation

$$\Delta(a_n(\Delta(b_n \Delta x_n))^\alpha) - p_n x_{\sigma(n)}^\beta = 0, \quad \beta > \alpha \quad (9)$$

has property B.

Corollary 7. *Let condition (4) holds and*

$$\limsup_{n \rightarrow \infty} \left\{ B_{\sigma(n)}^{-1} \sum_{s=N}^{n-1} B_{\sigma(s)} B_{s+1} P_s + \sum_{s=n}^{\sigma(n)-1} B_{s+1} P_s + B_{\sigma(n)} \sum_{s=\sigma(n)}^{\infty} P_s \right\} > 1$$

then the halflinear difference equation

$$\Delta(a_n(\Delta(b_n \Delta x_n))^\alpha) - p_n x_{\sigma(n)}^\alpha = 0 \quad (10)$$

has property B.

Corollary 8. *Let condition (4) holds and*

$$\limsup_{n \rightarrow \infty} \left\{ \frac{1}{B_{\sigma(n)}} \sum_{s=N}^{n-1} B_{\sigma(s)}^{\beta/\alpha} B_{s+1} P_s + \frac{B_{\sigma(n)}^{\beta/\alpha}}{B_{\sigma(n)}} \sum_{s=n}^{\sigma(n)-1} B_{s+1} P_s + B_{\sigma(n)}^{\beta/\alpha} \sum_{n=\sigma(n)}^{\infty} P_n \right\} > 0$$

then the sublinear difference equation

$$\Delta(a_n(\Delta(b_n \Delta x_n))^\alpha) - p_n x_{\sigma(n)}^\beta = 0, \quad \alpha > \beta \quad (11)$$

has property B.

4 Oscillation Results

In this section, we present oscillation criteria for equation (1).

Theorem 9. *Let condition (3) holds. Assume that*

$$\lim_{u \rightarrow \pm\infty} \frac{u}{f^{1/\alpha}(u)} = k_3 < \infty. \quad (12)$$

If

$$\limsup_{n \rightarrow \infty} \frac{1}{C_{\sigma(n)}} \sum_{s=N}^{\sigma(n)-1} \frac{1}{b_s} \sum_{t=s}^{n-1} \frac{1}{a_t^{1/\alpha}} \left[\sum_{j=t}^{n-1} p_j f(C_{\sigma(j)}) \right]^{1/\alpha} > k_3,$$

then the class $S_3 = \phi$ for equation (1).

Proof. Assume that equation (1) has an eventually positive solution $\{x_n\} \in S_3$ for $n \geq N$. Summing (1) from n to $s-1$ yields

$$(\Delta(b_s \Delta x_s))^\alpha \geq \frac{1}{a_s} \sum_{t=n}^{s-1} p_t f\left(\frac{x_{\sigma(t)}}{C_{\sigma(t)}} C_{\sigma(t)}\right) \geq f\left(\frac{x_{\sigma(t)}}{C_{\sigma(t)}}\right) \frac{1}{a_s} \sum_{t=n}^{s-1} p_t$$

Summing the last inequality twice in s , and then setting $s = \sigma(n)$ and $z_n = \frac{x_{\sigma(n)}}{C_{\sigma(n)}}$, we obtain

$$\frac{z_n}{f^{1/\alpha}(z_n)} \geq \frac{1}{C_{\sigma(n)}} \sum_{t=n}^{\sigma(n)-1} \left[\frac{1}{b_t} \sum_{j=t}^{n-1} \frac{1}{a_j^{1/\alpha}} \left(\sum_{i=j}^{n-1} p_i \right)^{1/\alpha} \right].$$

Taking $\limsup$ as $n \rightarrow \infty$ on both sides of the last inequality, we get a contradiction and this completes the proof. $\square$

By using S_1 and S_3 are empty we have the following theorem.

Theorem 10. *Let all conditions of Theorem 4 (Theorem 5) and Theorem 9 are hold. Then every solution of equation (1) is oscillatory.*

5 Examples

In this section we present three examples to illustrate the main results.

Example 5.1. *Consider the difference equation*

$$\Delta\left(\frac{1}{n}(\Delta^2 x_n)^{1/3}\right) - \frac{4^{1/3}(2n+1)}{n(n+1)} x_{n+4}^{1/3} = 0, \quad n \geq 1. \quad (13)$$

Simple computation show that $A_n \sim \frac{n^4}{4}$, $B_n \sim n$, $C_n \sim \frac{n^5}{20}$. Then it is easy to see that all conditions of Corollary 7 and Theorem 10 are satisfied and hence every solution of equation (13) is oscillatory. In fact $\{x_n\} = \{(-1)^n\}$ is one such oscillatory solution of (13).

Example 5.2. *Consider the difference equation*

$$\Delta\left(\frac{1}{n^3}(\Delta\left(\frac{1}{n}\Delta x_n\right))^3\right) - \frac{1}{n(n+1)} x_{n+4} = 0, \quad n \geq 1. \quad (14)$$

Simple computation show that $A_n \sim \frac{n^2}{2}$, $B_n \sim \frac{n^2}{2}$, $C_n \sim \frac{n^5}{10}$ and $p_n = \frac{1}{n(n+1)}$. It is easy to see that all conditions of Corollary 8 are hold and hence the equation (14) has property B.

Example 5.3. Consider the difference equation

$$\Delta\left(\frac{1}{n}\Delta\left(\frac{1}{n}\Delta x_n\right)\right) - \frac{1}{n(n+1)}x_{n+5}^3 = 0, \quad n \geq 1. \quad (15)$$

Simple computation show that $A_n = \frac{n(n-1)}{2}$, $B_n = \frac{n(n-1)}{2}$, $C_n = \frac{n(n-1)(3n^2-5n+1)}{6}$, $\rho_n = 1$. One can easily verify that all conditions of Corollary 6 are hold and hence equation (15) has property B.

Conclusion: The results established in this paper are of high generality and improve and complement earlier results known for special cases of equation (1).

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