I-CAUCHY SEQUENCE AND I-CORE OF A SEQUENCE IN ULTRAMETRIC FIELDS

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Abstract

In this paper, $K$ denotes a complete, locally compact, non-trivially valued, non-archimedean field. The entries of sequences, series and infinite matrices are in $K$. Here, we have defined I-Cauchy sequence and I of a sequence and proved a few theorems on I-Cauchy sequence and I-core. Also we have proved the inclusion relation between the $K$-core and the I-core of a sequence in such fields $K$.

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1 Introduction

The concept of I-convergence was introduced by P. Kostyrko, T. Salat and W. Wilezynski \cite{7} in classical analysis. Here we have defined I-convergent, I-Cauchy sequence and I-core of a sequence and proved a few theorems in non classical analysis.
Let
\[ Ax = (Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k, \quad n = 1, 2, 3, \ldots, \]
it being assumed that the series on the right converge and \((Ax)_n\) is called the \(A\)-transform of \(x = \{x_k\}\). The infinite matrix \(A = (a_{nk})\), \(a_{nk} \in K\), \(n, k = 1, 2, 3, \ldots\) is said to be regular if \((Ax)_n\) converges whenever \(x = \{x_k\}\) converges and have the same limit.

**Definition 1.** Let \(x = \{x_k\}\), \(x_k \in K\), \(k = 1, 2, 3, \ldots\) we denote by \(C_n(x)\), \(n = 1, 2, 3, \ldots\) the smallest closed \(K\)-convex set containing \(x_n, x_{n+1}, \ldots\) and call
\[ \mathcal{K}(x) = \bigcap_{n=1}^{\infty} C_n(x), \text{ the core of } x. \]

**Definition 2.** A non-empty subset \(I\) of a ring \(R\) of subsets of \(N \subset X\) is an ideal in \(R\) if and only if
(i) \(A, B \in I\) implies \(A \cup B \in I\),
(ii) \(A \in I\), \(B \in R\), \(B \subseteq A\) implies \(B \in I\).

An ideal is called non-trivial if \(N \notin I\).

**Definition 3.** A non-trivial ideal \(I\) in \(N\) is called admissible if \(\{x\} \in I\) for each \(x \in N\).

**Definition 4.** Let \(I\) be a non-trivial ideal. Then a sequence \(x = (x_k), x_k \in K, k \in N\), is said to be \(I\)-convergent to \(L \in K\) if for every \(\epsilon > 0\) the set \(\{k \in N : |x_k - L| \geq \epsilon\}\). In this case we write \(I - \lim_{k \to \infty} x = L\).

## 2 I-Cauchy Sequence

**Definition 5.** Let \(X\) be a non-archimedean normed space and \(I\) be an admissible ideal. Then a sequence \((x_k), x_k \in X\), is called an \(I\)-cauchy sequence in \(X\) if for every \(\epsilon > 0\),
\[ \{k \in N : \|x_{k+1} - x_k\| \geq \epsilon\} \in I. \]
Theorem 6. Let \( I \) be an admissible ideal. Then, 
\[ I - \lim_{k \to \infty} x_k = l \] 
implies that \((x_k)\) is an \( I \)-Cauchy sequence.

Proof. Let, \( I - \lim_{k \to \infty} x_k = l \). Then for each \( \epsilon > 0 \),
\[ A(\epsilon) = \{ k \in \mathbb{N} | \| x_k - l \| \geq \epsilon \} \in I \] (1)

For \( k + 1 \notin A(\epsilon) \), let us define
\[ A'(\epsilon) = \{ k \in \mathbb{N} | \| x_{k+1} - x_k \| \geq \epsilon \} \] (2)

Now to prove \( A'(\epsilon) \in I \).
Consider,
\[
\| x_{k+1} - x_k \| = \| (x_{k+1} - l) - (x_k - l) \| \\
\leq \max\{\| x_{k+1} - l \|, \| x_k - l \| \}
\Rightarrow \max\{\| x_{k+1} - l \|, \| x_k - l \| \} \geq \| x_{k+1} - x_k \| \\
\geq \epsilon \{\text{using (2)}\}
\]

Since
\[ k + 1 \notin A(\epsilon) \in I, \| x_{k+1} - l \| < \epsilon \] (4)

Therefore from (3) and (4), \( \| x_k - l \| \geq \epsilon \) implies that \( k \in A(\epsilon) \).
Hence \( A'(\epsilon) \subseteq A(\epsilon) \subseteq I \).
(i.e) \{ \( k \in \mathbb{N} : \| x_{k+1} - x_k \| \geq \epsilon \} \in I \) implies that \((x_k)\) is an \( I \)-Cauchy sequence.

Definition 7. A non-archimedean normed space \( X \) is said to be spherically complete if every nest of closed spheres has non-empty intersection.

Definition 8. Let \( X \) be a non-archimedean normed space and let the sequence \((x_k)\) of elements from \( X \) be such that there exists an \( k_0 \) such that whenever \( k_0 \leq k_1 < k_2 < k_3 \), then \( \| x_{k_3} - x_{k_2} \| < \| x_{k_2} - x_{k_1} \| \). Then \((x_k)\) is called a pseudo-cauchy sequence.

Definition 9. The element \( l \) is called a pseudo-limit of \((x_k)\) if for all \( k \) and \( m \) sufficiently large, \( k > m \) implies that \( \| x_k - l \| < \| x_m - l \| \), if the sequence \( (\| x_k - l \|) \) decreases monotonically.

Remark 10. The non-archimedean space \( X \) is spherically complete if and only if \( X \) is pseudo-complete.
**Theorem 11.**  
(i) If a non-archimedean normed space $X$ is spherically complete, then every $I$-cauchy sequence in $X$ is $I$-convergent in $X$.

(ii) If every $I$-cauchy sequence in $X$ is $I$-convergent in $X$, then $X$ is spherically complete.

**Proof.** (i) Let $(x_k)$ be an $I$-cauchy sequence in spherically complete space $X$.

$$A(\epsilon_k) = \{ k \in \mathbb{N} : \|x_{k+1} - x_k\| \geq \epsilon_k \} \in I.$$  

Consider the family of closed spheres $C_{\epsilon_1}(x_1) \supset C_{\epsilon_2}(x_2) \supset C_{\epsilon_3}(x_3) \supset \ldots$ 

Clearly $x_{k+1} \in C_{\epsilon_k}(x_k)$ for every $k$. Hence by spherical completeness,

$$\bigcap_{k \in K} C_{\epsilon_k}(x_k) \neq \emptyset$$

which implies that there must be some $l \in \bigcap_{k \in K} C_{\epsilon_k}(x_k)$ for every $k$.

Now to show that $I - \lim_{k \to \infty} x_k = l$, $l \in X$.

That is to show that

$$A(\epsilon) = \{ k \in \mathbb{N} : \|x_k - l\| \geq \epsilon \} \in I.$$  

To prove $A(\epsilon) \in I$, it is enough to prove $A(\epsilon) \subset A(\epsilon_k)$.

Now $k \in A(\epsilon)$ implies that $A(\epsilon) = \{ k \in \mathbb{N} : \|x_k - l\| \geq \epsilon \}$.

Consider,

$$\|x_k - l\| = \|x_k + x_{k+1} - x_{k+1} - l\|$$

$$= \|(x_{k+1} - l) - (x_{k+1} - x_k)\|$$

$$\leq \max\{\|x_{k+1} - l\|, \|x_{k+1} - x_k\|\}$$  

(5)

From (5) we have

$$\max\{\|x_{k+1} - l\|, \|x_{k+1} - x_k\|\} \geq \|x_k - l\| \geq \epsilon$$  

(6)

For $\epsilon > 0$, there exists $k + 1 \in \mathbb{N}$ such that $\epsilon_{k+1} < \epsilon$, we have

$$\|x_{k+1} - l\| \leq \epsilon_{k+1} < \epsilon$$
Therefore from (6) we conclude that $\|x_{k+1} - x_k\| > \epsilon$ which implies $k \in A(\epsilon_k)$.
Hence $A(\epsilon) \subset A(\epsilon_k) \in I$.

$$(i.e) \{k \in \mathbb{N} : \|x_k - l\| \geq \epsilon\} \in I.$$  

Therefore $(x_k)$ is $I$-convergent to $l$.

(ii) Assume every $I$-cauchy sequence in $X$ is $I$-convergent in $X$.
To prove $X$ is Spherically complete, it is enough to prove $X$ is pseudo-complete.
Let $x = (x_k), x_k \in X$, be an $I$-cauchy sequence which converges to $l, l \in X$.

$$A(\epsilon_k) = \{k \in \mathbb{N} : \|x_{k+1} - x_k\| \geq \epsilon_k\} \in I$$ \hfill (7)

and

$$\{k \in \mathbb{N} : \|x_k - l\| \geq \epsilon\} \in I$$ \hfill (8)

For $k < k + 1 < k + 2$, we have a nested closed spheres $C_{\epsilon_1}(x_1) \supset C_{\epsilon_2}(x_2) \supset C_{\epsilon_3}(x_3) \supset \ldots$.

From (7),

$$\|x_{k+1} - x_k\| \geq \epsilon_k > \epsilon_{k+1}$$ \hfill (9)

Let us now consider $\epsilon_{k+1} > 0$. Then there exists $k + 2 > k + 1 \in \mathbb{N}$, such that

$$x_{k+1} \in C_{\epsilon_{k+1}}(x_{k+1}),$$ which implies that $\|x_{k+2} - x_{k+1}\| < \epsilon_{k+1}$ \hfill (10)

From (9) and (10),

$$\|x_{k+2} - x_{k+1}\| < \epsilon_{k+1} < \|x_{k+1} - x_k\|$$
$$\Rightarrow \|x_{k+2} - x_{k+1}\| < \|x_{k+1} - x_k\|.$$  

Therefore, $(x_k)$ is a pseudo-cauchy sequence.

Now to prove $(x_k)$ has a pseudo-limit $l$.

Now let us take $l \in C_{\epsilon_r}(x_r)$. Therefore, we have

$$\|x_r - l\| < \epsilon_r < \epsilon_k$$ for $r > k$. \hfill (11)
But from (8),
\[ \|x_k - l\| \geq \epsilon_k \tag{12} \]
From (11) and (12),
\[ \|x_r - l\| < \|x_k - l\| \text{ for } r > k. \]
Hence \( l \) is a pseudo-limit of \((x_k)\) which implies that \( X \) is pseudo-complete.
This completes the proof. \( \square \)

# 3 I-core of a Sequence

**Definition 12.** Let \( I \) be an admissible ideal. For any sequence \( x = (x_k), x_k \in K, \) \( k = 1, 2, 3, \ldots \) For each \( u \in K \), the \( I \)-core of \( x \) is given by
\[
I\text{-core}(x) = \bigcap_{u \in K} B_x(u), \text{ where } \\
B_x(u) = \{ w \in K : |w - u| \leq I - \lim_{k \to \infty} \sup_k |x_k - u| \}. 
\]

**Remark 13.** In the definition of the \( K(x) \), the closed \( K \)-convex set \( C_n(x) \) is the intersection of all closed disks that contain \( x = (x_k), x_k \in K, k = 1, 2, 3, \ldots \) in defining the \( I \)-core \( (x_k) \) we have replaced the sequence \( (x_k) \) by an arbitrary subsequence in which \( \{k \in \mathbb{N} : x_k \text{ belongs to the closed disk} \} \notin I \). Therefore, it is clear that
\[
I\text{-core}(x) \subset K\text{-core}(x) \text{ for all } x. 
\]

**Theorem 14.** Let \( I \) be an admissible ideal and let \( \{k \in \mathbb{N} : |x_k - u| < r + \epsilon \} \notin I \) where \( r = I - \lim_{k \to \infty} \sup_k |x_k - u| \). For any bounded sequence \( x = (x_k) \), if the matrix \( A = (a_{nk}), a_{nk} \in K \) satisfies the following conditions:

(i) \( A \) is regular,

(ii) \( \lim_{n \to \infty} \sup_{k \geq 1} |a_{nk}| = 1, \)

then \( K\text{-core}(Ax) \subset I\text{-core}(x). \)
Proof. Assume (i) and (ii) hold.
To prove $K\text{-core}(Ax) \subset I\text{-core}(x)$
Let $w \in K\text{-core}(Ax)$, for any point $u \in K$, we have

$$|w - u| \leq \limsup_{n \to \infty} \sup_{k \geq 1} \left| \sum_{k=1}^{\infty} a_{nk}x_k - u \right|$$

$$= \limsup_{n \to \infty} \left| \sum_{k=1}^{\infty} a_{nk}x_k - \sum_{k=1}^{\infty} a_{nk}u \right|$$

since $A$ is regular

$$= \limsup_{n \to \infty} \left| \sum_{k=1}^{\infty} a_{nk}(x_k - u) \right|$$

(13)

Given $r = I - \limsup_{k \to \infty} |x_k - u|$ and let $E = \{k \in \mathbb{N} : |x_k - u| < r + \epsilon\}$.
Then $E \notin I$ and we have

$$\left| \sum_{k=1}^{\infty} a_{nk}(x_k - u) \right| \leq |a_{n1}(x_1 - u) + a_{n2}(x_2 - u) + \ldots|$$

$$\leq \max\{|a_{n1}(x_1 - u)|, |a_{n2}(x_2 - u)|, \ldots\}$$

$$\leq \max\{|a_{n1}|x_1 - u| + |a_{n2}|x_2 - u| + \ldots\}$$

$$\leq \sup_{k \geq 1} |a_{nk}|x_k - u|$$

$$\limsup_{n \to \infty} \left| \sum_{k=1}^{\infty} a_{nk}(x_k - u) \right| \leq |x_k - u| \quad \text{using (ii)}$$

$$< r + \epsilon$$

$$\leq I - \limsup_{k \to \infty} |x_k - u|$$

From (13), $|w - u| \leq I - \limsup_{k \to \infty} |x_k - u|$ for $E \notin I$.
Therefore, $w \in I\text{-core}(x)$.
Hence, $K\text{-core}(Ax) \subset I\text{-core}(x)$.
This completes the proof of the theorem.
References


