Rainbow Vertex Coloring for Shadow, Square and Spilting Graph of Bi-star Graph

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Abstract

A vertex-connected graph \(G\) is rainbow vertex-connected if any two vertices are connected by a path whose internal vertices have distinct colors. A vertex-coloring under which \(G\) is rainbow vertex-connected is called a rainbow vertex-coloring. The rainbow vertex-connection number of a connected graph \(G\) denoted by \(rvc(G)\), is the smallest number of colors that are needed in order to make \(G\) is Rainbow vertex-connected. The transformation of ideas of Rainbow connection was introduced by Chartrand et al [1]. In this paper we endeavour to bring together most of the results and papers that dealt with it.

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Key Words and Phrases: Rainbow Path, Rainbow vertex connection number, Shadow graph, Square graph, Spilting graph, Bi-star Graph.
1 Introduction

In this paper, all graphs are finite, simple and undirected. Any notation or terminology not defined here, follows that used in [2]. For a graph G, let $V(G)$, $E(G)$, $n(G)$ and $m(G)$ be the set of vertices, the set of edges, the order and size of G respectively. The most fundamental graph-theoretic subject, both in a combinatorial sense and an algorithmic sense is connectivity. In graph theory, connectivity exhibits refined and powerful results. There are many options for the effectiveness on connectivity concept, such as hamiltonicity, k-connectivity and so on. There is another interesting way to strengthen the connectivity requirement, the rainbow connection, was introduced by chartrand et al [1] in 2008.

There are many Government agencies for the purpose of communicating the information related to national security. They communicate with each other, through their regular channels, from radio systems to databases. After the terrorist attack held, they ascertained that there is a weaknesses in the transfer of classified information. They realized the weaknesses are that the law enforcement and intelligence agencies are not able to communicate each other through their regular channels from radio systems to databases. There are many distinct technologies are utilized for the communication purpose and prohibited shared access (there is no way to cross check the information between various organizations). The two-fold issues are, one is the information needs to be protected since it relates to national security and the other is there must be many procedures to permit access between appropriate parties. Between any two agencies, there will be intermediate agencies in which the information passes through one or more secure paths with large number of passwords required, which cannot be accessed by the intruders. Moreover, the passwords should not be repeated in the desirable path. Providing minimum number of passwords, while transfer the classified information through intermediate agencies, so that the passwords along each path should be different. This situation can be modeled by graph-theoretic model. Let G be nontrivial connected Graph on which on which an edge Coloring mapping positive integers is defined, where adjacent edges may be colored the same. A path is rainbow if no two edges of it are colored the same. An edge coloring Graph G is Rainbow connected.
if every two vertices are connected by rainbow path. In a rainbow coloring, we need only find one rainbow path connecting every two vertices. A vertex-connected graph $G$ is rainbow vertex-connected if any two vertices are connected by a path whose internal vertices have distinct colors. A vertex-coloring under which $G$ is rainbow vertex-connected is called a rainbow vertex-coloring. The rainbow vertex-connection number of a connected graph $G$ denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make $G$ is rainbow vertex-connected. Finally $rvc(G) = 0$ iff $G$ is a clique. In rainbow vertex coloring, we assign colors to the vertices as $\{1, 2, 3, \cdots, k\}$. This paper is mainly concerned with rainbow vertex coloring and rainbow vertex connection number of Shawdow, Square and Spilt graph of Bi-star Graph.

**Definition 1.** Let $G$ be a graph with $V(G) = \{v_1, v_2, \cdots, v_n\}$. The shadow graph $Shad(G)$ of $G$ is that graph with vertex set $V(G) \cup \{u_1, u_2, \cdots, u_n\}$ where $u_i$ is called the shadow vertex of $v_i$ and $u_i$ is adjacent to $u_j$ if $v_i$ is adjacent to $v_j$ and $u_i$ is adjacent to $v_j$ if $v_i$ is adjacent to $v_j$ for $1 \leq i, j \leq n$.

**Definition 2.** The square $G^2$ of a graph $G$ has $V(G^2) = V(G)$ with $u,v$ adjacent in $G^2$ whenever $d(u, v) \leq 2$ in $G$.

**Definition 3.** For each vertex $v$ of a graph $G$, take a new vertex $v'$. Join $v'$ to all the vertices of graph $G$ adjacent to $v$. The graph $S(G)$ thus obtained is called Splitting Graph of $G$.

**Definition 4.** A Variant of the Randic index of a graph $G$, denoted by $R'(G)$ where $R'(G) = \sum_{u,v \in E(G)} \frac{1}{\max\{d(u), d(v)\}'}$, and $d(u)$ denotes the degree of a vertex $u$ in $G$.

## 2 Preliminaries

**Theorem 1.**
A connected graph $G$ with $n$ vertices has $rvc(G) < \frac{11n}{\delta(G)}$.

**Theorem 2.**
Let $G$ be a 2-connected graph of order $n (n \geq 3)$. Then $rvc(G) = 0$ if $n = 3$; $1$ if $n = 4, 5$ and $rvc(G) = 3$ if $n = 9$; $\left\lceil \frac{n}{2} \right\rceil$ if $n =$
Proposition 1. Let \( G = M[K_{1,n}] \) then \( rvc(G) = \chi(G) - 1 = \alpha(G) - 1 = \omega(G) - 1 \).

Proposition 2. Let \( G = M[K_{1,n}] \) without isolated vertices then \( rvc(G) = \gamma(G) \).

Proposition 3. Let \( G = M[K_{1,n}] \) on \( n \) vertices with \( q \) vertices of degree atleast 2 then \( rvc(G) < n - 1 + q \).

Proposition 4. Let \( G = M[K_{1,n}] \) then \( rvc(G) < \lfloor \frac{n + \omega(G)}{2} \rfloor \).

Proposition 5. Let \( G = M[K_{1,n}] \) then \( rvc(G) = \lfloor \frac{n}{2} \rfloor + \left\lceil \frac{n}{4} \right\rceil - \lfloor \frac{n}{4} \rfloor - 1 \).

Proposition 6. Let \( G = M[K_{1,n}] \) then \( rvc(G) = \left\lfloor \frac{n}{2} \right\rfloor \) if \( n = 2k + 1, k \geq 1 \).

Main Theorems:

Theorem 3. Let \( G = \text{shad}(B_{n,n}), n \geq 2 \) then \( rvc(G) = \frac{a_1 + b_1}{\Delta(G)} \) where \( |X| = a_1 \) and \( |Y| = b_1 \), \( X = \{v : \text{deg}_G(v) = \Delta(G)\} \) and \( Y = \{v : \text{deg}_G(v) = \delta(G) = 2\} \).

Proof. For simplicity, we write \( G = \text{shad}(B_{n,n}) \), of order \( 4(n + 1) \). Vertex set \( V(G) = \{u, v\}, U \{x, y\}, U \{u_i, v_i, x_i, y_i : 1 \leq i \leq \frac{\Delta(G)}{2} - 1\} \), clearly \( |V(G)| = 4(n + 1) = 2\Delta(G) \) and \( E(G) = \{uv\} \cup \{uv\} \cup \{vx\} \cup \{xy\} \cup \{uu_i, vv_i, ux_i, vy_i, yy_i, xu_i : 1 \leq i \leq \frac{\Delta(G)}{2} - 1\} \), and \( |E(G)| = 4(\Delta(G) - 1) \). Define a vertex coloring \( c : V(G) \rightarrow \left\{1, \frac{a_1 + b_1}{\Delta(G)}\right\} \) as follows.

\[ c(u_i) = c(x_i) = c(v_i) = c(y_i) = 1 \text{ for every } i \left(1 \leq i \leq \frac{\Delta(G)}{2} - 1\right) \]
and \( c(u) = c(x) = c(v) = 1 \) and \( c(y) = \frac{a_1 + b_1}{\Delta(G)} \). Every path is rainbow vertex connected with this coloring.

Thus \( rvc(G) = \frac{a_1 + b_1}{\Delta(G)} \). Suppose \( rvc(G) \neq \frac{a_1 + b_1}{\Delta(G)} \). First we take \( rvc(G) < \frac{a_1 + b_1}{\Delta(G)} \) then assigning the colors to the vertices fewer than \( \frac{a_1 + b_1}{\Delta(G)} \) colors then some colors are repeated in the some paths, which is not rainbow vertex connected. This case fails.

Suppose if we take \( rvc(G) \geq \frac{a_1 + b_1}{\Delta(G)} \) colors, then every path is rainbow vertex connected with maximum colors chosen. Thus taking \( \frac{a_1 + b_1}{\Delta(G)} \) colors will be minimum in order to take rainbow vertex connected.

**Theorem 4.**

If \( v \) is a vertex of Graph \( G \) where \( \text{deg}_G(v) = \delta(G) \), then \( R'(G) - rvc(G) > 0 \) where \( G = \text{shad}(B_{n,n}) \).

**Proof.** Since \( \text{deg}_G(v) = \delta(G) = 2 \), then \( |N(v)| = 2 \), let \( X = \{v : \text{deg}_G(v) = \delta(G)\} \) and \( Y = \{v : \text{deg}_G(v) = \delta(G)\} \) and \( |X| = a_1 \) and \( |Y| = b_1 \) obviously \( a_1 + b_1 = |V(G)| \). It is easy to verify that

\[
R'(G) - rvc(G) = \sum_{u,v \in E(G)} \frac{1}{\max\{d(u), d(v)\}} - \frac{a_1 + b_1}{\Delta(G)} > 0
\]

**Theorem 5.**

Let \( G = \text{shad}(B_{n,n}) \) be a connected graph of order \( n \) and \( \text{diam}(G) = 3 \). If \( \alpha(G) = n - 4 \) and \( X \) is a maximum independent set of \( G \), then \( rvc(G) = 2 \) if \( |V(G) - X| \cong C_4 \).

**Proof.** Let \( H = |V(G) - X| \) and \( c \) be a vertex coloring of \( G \). Since \( \text{diam}(G) = 3 \). Hence \( rvc(G) = \text{diam}(G) - 1 \). We prove this case by analyzing the structure of graph \( H \). Since \( H \) is connected \( H \cong C_4 \). Let \( V(H) = \{u, y, v, x\} \). Define a vertex coloring \( c \) as follows:

\[
c(v) = 2 \text{ for } v \in \left[U_{1 \leq i \leq \frac{\Delta(G)}{2} - 1}N(v_i)\right] \cap \left[U_{1 \leq i \leq \frac{\Delta(G)}{2} - 1}N(y_i)\right] - \{y\}
\]

and color all other vertices with 1. It is easy to verify that every path is rainbow vertex connected graph \( G \). Thus \( rvc(G) = 2 \).
Theorem 6.
Let \( G = (B^2_{n,n}) \) then \( rvc(G) = 1 \).

Proof. For simplicity, we write \( G = (B^2_{n,n}) \) the vertex set \( V(G) = \{u, v\} \cup V_1 \cup V_2 \), where \( V_1 = \{u_1, \cdots u_{\delta(G)-1}\} \) and \( V_2 = \{v_1, \cdots v_{\delta(G)-1}\} \) and \( |V(G)| = 2\delta(G) \) and the edge set \( E(G) = \{uv\} \cup \{uu_i, vv_i, vu_i, uv_i : i \in [1, \delta(G)-1]\} \) then \( |E(G)| = 2\Delta(G) + 1 \).

Define a coloring \( c : V(G) \rightarrow \{1\} \) as follows:
\[
c(u_i) = 1; 1 \leq i \leq \delta(G) - 1
\]
\[
c(v_i) = 1; 1 \leq i \leq \delta(G) - 1
\]
\[
c(u) = c(v) = 1.
\]
path is rainbow vertex connected with this coloring. Thus \( rvc(G) = 1 \). \( \square \)

Theorem 7.
Let \( G = S(B_{n,n}) \) then \( rvc(G) = 2 \).

Proof. For simplicitely, we write \( G = S(B_{n,n}) \), the vertex set \( V(G) = \{u, v, x, y\} \cup \{u_i, v_i, x_i, y_i : i \in \left[ \frac{\Delta(G) - \delta(G)}{2} \right] \} \)
and \( |V(G)| = 4 \left[ 1 + \left[ \frac{\Delta(G) - \delta(G)}{2} \right] \right] \) and edge set is
\[
E(G) = \{uv, uy, xv\} \cup \{uu_i, ux_i, xu_i, vv_i, vy_i, yv_i : i \in \left[ \frac{\Delta(G) - \delta(G)}{2} \right] \}
\]
\( |E(G)| = 3(\Delta(G) - \delta(G)) \).

Define a vertex coloring \( c : V(G) \rightarrow \{1, 2\} \) as follows:
\[
c(u_i) = c(v_i) = c(x_i) = c(y_i) = 1 \text{ where } i \in \left[ 1, \left[ \frac{\Delta(G) - \delta(G)}{2} \right] \right]
\]
and \( c(u) = c(x) = c(y) = 1, c(v) = 2 \).
Every path is rainbow vertex connected with this coloring. Thus \( rvc(G) = 2 \). \( \square \)

3 Conclusion:
The rainbow vertex connection number for the shadow graph of Bi-star Graph is maximum degree of a graph G divides the order of a graph G and also the difference between randic index and rainbow vertex connection number is always non-negative. The rainbow vertex connection number for square graph of Bi-star Graph is 1.
where as rainbow vertex connection number for spilt graph of Bi-
star Graph is 2.

4 Applications:

Its application is to secure transfer of classified information between
agencies and it also be stimulated by its interesting elucidation in
the area of networking. The information needs to be protected
since it relates to national security; there must also be procedures
that permit access between adequate parties. This twofold issue
can be addressed by assigning information transfer paths between
agencies which have many other agencies as intermediaries while
requiring large enough number of passwords and firewalls that is
prohibitive to intruders. The minimum number of passwords or
firewalls needed that allows one or more paths between every two
agencies so that the passwords along each path are distinct. Sup-
pose that G is represents a network. We wish to route messages
between any two vertices in a pipeline and require that each link
on the route between the vertices. Clearly, we want to minimize
the number of distinct channels that we use in our network.

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