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## A Review of Fuzzy Fractional Differential Equations

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### Abstract

Fuzzy fractional differential equations play an important role in modelling of science and engineering problems. This paper present a review on recent development of fuzzy fractional differential equations and a survey of analytical and numerical methods to solve fuzzy fractional differential equations proposed by several authors is elucidated in detail.

**Keywords:** Fuzzy number, Fuzzy fractional differential equation, Caputo H-differentiability, Riemann-Lieouville H-differentiability.

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## 1 Introduction

Fractional calculus is a generalization of differentiation and integration to an arbitrary order. In recent years fractional differential equations have attracted a considerable interest both in mathematics and its applications. Many recently developed models in areas like control[12], viscoelasticity[21], signal processing[40], electrochemistry, diffusion processes, etc. are formulated in terms of fractional derivatives or fractional integrals. The history and properties of fractional differential equations have been given by Podlubny[29], Kilbas[20] and Miller and Ross[23].

In the past three decades, many researchers had contributed to fuzzy fractional differential equations. Agarwal et al.[2] proposed the concept of solution for the fractional differential equations with uncertainty. R.P.Agarwal is a pioneer of the fuzzy fractional differential equations. The author have considered the Riemann-Liouville differentiability with a fuzzy initial conditions to solve fuzzy fractional differential equations. Sadia Arshad et al.[37] introduced the fractional differential equations with uncertainty and provided some results on the existence and uniqueness of solutions of fuzzy fractional differential equations.

Consequently, Djurdjica Takaci et al.[15] analysed the fuzzy fractional differential equations with fuzzy coefficients. The fuzzy fractional derivatives are considered in the Caputo sense and Hukuhara difference. The authors obtained the exact and approximate solutions by the fuzzy Mikusinski operators and this solutions are visualized by the package GeoGebra for different values of  $\beta_1$  and  $\beta_2$  in the figures (where are order of fractional derivative and  $0 < \beta_1 < 1$  and  $0 < \beta_2 < 1$ )

Also Djurdjica Takaci et al.[16] used Fuzzy Laplace Transform method to solve fuzzy fractional differential equations with fuzzy initial conditions under Caputo Hukuhara differentiability.

Prakash et al.[30] analysed the initial value problems for differential equations of fractional order with uncertainty. They have used predictor-corrector method for solving fuzzy fractional initial value problem. They compared the numerical solutions with the corresponding exact solution through illustrated problems and also they showed that the order of convergence of the predictor-correct method is  $\min[2, 1 + \beta]$  (where  $\beta$  is order of fractional derivate)

Further, Abbasbandy et al.[1] introduced the new type of fuzzy fractional differential equations known as fuzzy local fractional differential equations under fuzzy Riemann-Liouville differentiability. They also provided some new results such as the relation between each type of fuzzy local fractional differentiability and their r-cuts, the composition of the fuzzy KG-LFD and related fuzzy integrals.

In addition, Allahviranloo et al.[9] obtained the explicit solutions of the fuzzy linear fractional differential equations under Riemann-Liouville H-differentiability according to the equivalent Volterra integral. The authors solved this Volterra integral by successive approximation method and using this they solved homogeneous and nonhomogeneous nuclear decay equations with decay parameter  $\lambda$  as illustrated problems.

The existence, uniqueness of approximate solutions for fuzzy fractional differential equations under Caputos H-differentiability presented by Salahshor et al.[39]. They transformed fuzzy fractional differential equation into equivalent form of fuzzy Volterra integral equation.

Arshad[11] established equivalence between fuzzy fractional differential equation and fuzzy integral equation. Also by applying this integral equation, the author provided uniqueness and existence of solutions of fuzzy fractional differential equation. The author used Goetschel-Voxman derivative.

Recently Armand et al.[10] provided the existence and uniqueness solutions of fuzzy fractional differential equation by contraction mapping principle and the fixed point theorem.

In section 2 we provided some preliminaries in connection with fuzzy fractional differential equations. In section 3 the methods for solving fuzzy fractional differential equations are discussed in detail.

## 2 Preliminaries

L.A.Zadeh[44] introduced fuzzy sets and established properties of fuzzy sets. The basic definitions of fuzzy number and definitions of the Riemann-Liouville and Caputo fractional derivatives of fractional calculus are given in [42].

### 2.1 Fuzzy Number [42]

A fuzzy number  $\tilde{U}$  is convex, normalized fuzzy set  $\tilde{U}$  of the real line  $\mathbb{R}$  such that  $\mu_{\tilde{U}}(x) : \mathbb{R} \rightarrow [0, 1], \forall x \in \mathbb{R}$  where  $\mu_{\tilde{U}}$  is called the membership function of the fuzzy set, and it is piecewise continuous. we define different types of fuzzy number as follows.

### 2.2 Triangular Fuzzy number (TFN)[42]

Let us consider an arbitrary TFN  $\tilde{U}=(a,b,c)$ . The membership  $\mu_{\tilde{U}}$  of  $\tilde{U}$  is defined as follows:

$$\mu_{\tilde{U}}(x) = \begin{cases} 0, & x \leq a \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ \frac{c-x}{c-b}, & b \leq x \leq c \\ 0, & x \geq c \end{cases}$$

### 2.3 Trapezoidal Fuzzy Number TrFN [42]

We consider an arbitrary trapezoidal fuzzy number (TrFN)  $\tilde{U}=(a,b,c,d)$ . The membership  $\mu_{\tilde{U}}$  of  $\tilde{U}$  is defined as follows:

$$\mu_{\tilde{U}}(x) = \begin{cases} 0, & x \leq a \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & b \leq x \leq c \\ \frac{d-x}{d-c}, & c \leq x \leq d \\ 0, & x \geq d \end{cases}$$

For all type of fuzzy numbers, the lower and upper bounds of the fuzzy numbers satisfy the following requirements:

- (i)  $\underline{u}(r)$  is a bounded left-continuous nondecreasing function over  $[0,1]$ ;
- (ii)  $\overline{u}(r)$  is a bounded right-continuous nonincreasing function over  $[0,1]$ ;
- (iii)  $\underline{u}(r) \leq \overline{u}(r)$ ,  $0 \leq r \leq 1$

### 2.4 Definition[42]

Let  $F : (a, b) \rightarrow \mathfrak{R}_F$  and  $t_0 \in (a, b)$ .  $X$  is called differentiable at  $t_0$ , if there exists  $F'(t_0) \in \mathfrak{R}_F$  such that

- (i) For all  $h > 0$  sufficiently close to 0, the Hukuhara difference  $F(t_0 + h) \ominus F(t_0)$  and  $F(t_0) \ominus F(t_0 - h)$  exists and (in metric D)

$$\lim_{h \rightarrow 0^+} \frac{F(t_0 + h) \ominus F(t_0)}{h} = \lim_{h \rightarrow 0^+} \frac{F(t_0) \ominus F(t_0 - h)}{h} = F'(t_0)$$

(ii) For all  $h > 0$  sufficiently close to 0, the Hukuhara difference  $F(t_0) \ominus F(t_0 + h)$  and  $F(t_0 - h) \ominus F(t_0)$  exists and (in metric D)

$$\lim_{h \rightarrow 0^+} \frac{F(t_0) \ominus F(t_0 + h)}{-h} = \lim_{h \rightarrow 0^+} \frac{F(t_0 - h) \ominus F(t_0)}{-h} = F'(t_0)$$

where  $\ominus$  is the Hukuhara difference and  $\mathfrak{R}_F$  is the class of fuzzy subsets of real axis.

## 2.5 Theorem [14]

Let  $F : (a, b) \rightarrow \mathfrak{R}_F$  and denote  $[F(t; r)] = [\underline{f}(t; r), \bar{f}(t; r)]$  for each  $r \in [0, 1]$

(i) If  $F$  is differentiable of the first type (I), then  $\underline{f}(t; r)$  and  $\bar{f}(t; r)$  are differentiable functions, and we have  $[F'(t; r)] = [\underline{f}'(t; r), \bar{f}'(t; r)]$

(ii) If  $F$  is differentiable of the second type (II), then  $\underline{f}(t; r)$  and  $\bar{f}(t; r)$  are differentiable functions, and we have  $[F'(t; r)] = [\bar{f}'(t; r), \underline{f}'(t; r)]$

proof: The proof of the theorem is given in Chalco-Cano and Roman-Flores [14].

## 2.6 Riemann-Liouville Fractional Integral [29]

The Riemann-Liouville integral operator  $J^\alpha$  of order  $\alpha > 0$  is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, t > 0 \quad (1)$$

## 2.7 Fuzzy Riemann-Liouville Fractional Integral [22]

The Riemann-Liouville fractional integral of order  $\alpha$  of fuzzy number valued function  $f$ , based on its  $r$ -cut representations, can be expressed as

$$[J^\alpha \tilde{f}(t; r)] = [J^\alpha \underline{f}(t; r), J^\alpha \bar{f}(t; r)], t > 0 \quad (2)$$

where

$$J^\alpha \underline{f}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \underline{f}(\tau) d\tau, t > 0 \quad (3)$$

$$J^\alpha \bar{f}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \bar{f}(\tau) d\tau, t > 0 \quad (4)$$

## 2.8 Caputo Derivative [29]

The fractional derivative of  $f(t)$  in the Caputo sense is defined as follows:

$$D^\alpha f(t) = J^{m-\alpha} D^m f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t - \tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, m-1 < \alpha < m \\ \frac{d^m}{dt^m} f(t), \alpha = m, m \in \mathbb{N} \end{cases} \quad (5)$$

Some basic properties of the fractional operator are as follows:

$$(i) J^\alpha J^\gamma f(t) = J^{\alpha+\gamma} f(t), \alpha, \gamma \geq 0 \quad (6)$$

$$(ii) J^\alpha(t^\gamma) = \frac{\Gamma(\gamma+1)t^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)}, \alpha > 0, \gamma > -1, t > 0 \quad (7)$$

## 2.9 Caputo-Type Fuzzy Fractional Derivatives [22]

Let  $\tilde{f}(t; r)$  be a fuzzy valued function and  $[\tilde{f}(t; r)] = [\underline{f}(t; r), \bar{f}(t; r)]$ , for  $r \in [0, 1]$ ,  $0 < \alpha < 1$  and  $t \in (a, b)$

(a) If  $\tilde{f}(t; r)$  is a Caputo-type fuzzy differential function in the first form, then

$$[D^\alpha \tilde{f}(t; r)] = [D^\alpha \underline{f}(t; r), D^\alpha \bar{f}(t; r)] \quad (8)$$

(b) If  $\tilde{f}(t; r)$  is a Caputo-type fuzzy differential function in the second form, then

$$[D^\alpha \tilde{f}(t; r)] = [D^\alpha \bar{f}(t; r), D^\alpha \underline{f}(t; r)] \quad (9)$$

where

$$D^\alpha \underline{f}(t; r) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \underline{f}^m(\tau) d\tau, m-1 < \alpha < m \quad (10)$$

$$D^\alpha \bar{f}(t; r) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \bar{f}^m(\tau) d\tau, m-1 < \alpha < m \quad (11)$$

$$\frac{d^m}{dt^m} f(t), \alpha = m, m \in \mathbb{N} \quad (12)$$

## 2.10 Fractional Initial Value Problem [29]

Let us consider the following fractional initial value problem

$$D^\alpha y(t) = f(t, y) \quad (13)$$

subject to the initial condition

$$y(0) = y_0, t \in [a, b], \alpha \in (0, 1), \quad (14)$$

where  $D^\alpha$  denotes the Caputo fractional differential operator.

## 2.11 Fuzzy Fractional Initial Value Problem (FFIVP) [22]

Let us consider the following fuzzy fractional initial value problem

$$D^\alpha \tilde{y}(t) = f(t, \tilde{y}), \quad (15)$$

subject to the initial condition

$$\tilde{y}(0) = \tilde{y}_0, t \in [a, b], \alpha \in (0, 1), \quad (16)$$

The FFIVP can be considered equivalently by the following initial value problems:

$$[D^\alpha \underline{y}(t), D^\alpha \overline{y}(t)] = [\underline{f}(t, \underline{y}), \overline{f}(t, \overline{y})] \quad (17)$$

subject to the fuzzy initial condition

$$[\underline{y}(0), \overline{y}(0)] = [\underline{y}_0, \overline{y}_0] \quad (18)$$

### 3 Methods applied to solve fuzzy fractional differential equation

#### 3.1 Operational Matrix Method

Ali Ahmadian et al.[4] have solved linear and nonlinear fuzzy fractional differential equation by the Operational matrix based on Legendre polynomials involving Caputo fractional derivative with fuzzy initial conditions. The author demonstrated the validity and efficiency of the proposed method through illustrations. Also they calculated absolute errors for different values and  $m$ . (where is order of fractional derivative and  $m$  denotes number of Legendre functions).

Also Ali Ahmadian[5] used shifted Jacobi polynomials in Operational matrix for solving fuzzy fractional differential equations under Caputo fractional derivative. In this method initial conditions are assumed by fuzzy number. The advantage of this method is that the matrix operators have the main role to find the approximate fuzzy solution of fuzzy fractional differential equations. The authors obtained fuzzy residual of fuzzy fractional differential equation. Also they stated that this method consume less time and cost for computation work.

#### 3.2 Zadehs Extension Method

M.Z.Ahmaed et al.[3] have proposed the study of fuzzy fractional differential equation and presented its solution using Zadehs extension principle. Initial condition is consider as a fuzzy number. In this article, they obtained the solution of fractional differential equation by Laplace transform and they fuzzified the solution of this equation using Zadehs extension principle. Also the author solved this fuzzy fractional differential equation using fractional Euler method. They presented algorithms for fuzzy fractional Euler method of linear and nonlinear problems. They showed when the fractional order approaches the integer order, the solution of fuzzy fractional differential equations approaches the solution of fuzzy differential equation through illustration.

#### 3.3 Fuzzy Sumudu Transform Method

Fuzzy Sumudu transform method was introduced by Norazrizal Aswad et al.[26] to solve fuzzy fractional differential equations with fuzzy initial conditions. The fractional derivatives are assumed by Caputo sense. Sumudu transform is one of the integral transform. In this article, the authors introduced a new property of Fuzzy Sumudu transform for Caputo fuzzy fractional derivative. They

provided a procedure of Fuzzy sumudu transform for solving fuzzy fractional differential equations. The authors concluded that the solution of fuzzy fractional differential equation approaches to the solution of fuzzy differential equation as  $\alpha$  approaches to 1 (where  $\alpha$  is order of fractional derivative).

### 3.4 Fuzzy Laplace Transform Method

Another integral transform is fuzzy Laplace transform. Several authors [38,36,43,8] used this method to solve fuzzy fractional differential equation. Salashour et al. [38] have obtained analytical solution of fuzzy fractional differential equation by fuzzy Laplace transform. In this article they considered Riemann-Liouville H-differentiability to solve fuzzy fractional differential equation. This fuzzy Laplace transform method reduce the fuzzy fractional differential equation to an algebraic equation. This Fuzzy Laplace transform method was used to solve fuzzy fractional differential equations under Riemann-Liouville H-differentiability by Rubanraj et al. [36]. The main advantage of this method is to solve the problem directly without finding a general solution. The author showed efficiency and utility of proposed method through illustration.

Salahshour et al. [43] applied the concept of Caputo's H-differentiability to solve the fuzzy fractional differential equation with uncertainty. They employed the fuzzy Laplace transform method to find analytical solutions of fuzzy fractional differential equations. For the first time in the literature, they solved fuzzy fractional Basset equation which is a mixed type of fractional differential equation under uncertainty. In this article, they solved some real-world problems such as nuclear decay equation and Basset problem.

Allahviranloo et al. [8] presented the fractional Green's functions for the solutions of fuzzy fractional differential equations by the fuzzy Laplace transforms under Caputo H-differentiability. They obtained solutions of classical fuzzy harmonic oscillator equation and fuzzy relaxation equations by fuzzy Laplace method.

### 3.5 Differential Transform Method

The differential transform method is based on Taylor series which contracts an analytical solution in the form of polynomials. This method is an iterative procedure for obtaining the analytical solution. Few authors [27,28,19,31] used differential transform method to solve fuzzy fractional differential equations. Osama et al. [27] have solved fuzzy fractional initial value problems by differential transform method. They showed validity and accuracy of the differential transform method through illustration by giving different values of  $r$ -level set.

Also Osama et al. [28] applied differential transform method for solving fuzzy fractional boundary value problems. The fuzzy number appeared in the boundary conditions. Using this differential transform method, fuzzy fractional heat equation was also solved by Ghazanfari et al. [19]. The derivative is considered as Caputo sense. Initial conditions are assumed by fuzzy number. In this article, the authors derived the generalized two-dimensional differential transform

method.

In[31], a new method generalized differential transform method for solving a class of fuzzy fractional differential equations was introduced by Rivaz et al. In this method, fuzzy fractional derivative are considered in the sense of the Caputo fractional derivative. They have computed derivatives by iterative procedure instead of calculating derivatives directly.

### 3.6 Homotopy Analysis Transform Method

Ahmed Salah et al.[6] have proposed a new method homotopy analysis transform (Homotopy analysis method and Laplace transform method) to obtain approximate solution of fuzzy fractional heat and wave equation. Here, the initial conditions are assumed by fuzzy number. The effectiveness of homotopy analysis transform method is elucidated in details through illustration by the authors.

### 3.7 Eigenvalue-Eigenvector Method

Eigenvalue-Eigenvector method proposed by Balooch et al. [13] for solving system of fuzzy fractional differential equations under fuzzy Caputo differentiability with fuzzy initial conditions. In this paper, they discussed all form of eigenvalues such as real and distinct eigenvalues, complex eigenvalues and multiple eigenvalues.

### 3.8 Modified Fractional Euler Method

Mehran Mazandarani et al.[22] used modified fractional Euler method to solve fuzzy fractional initial value problem. They applied predictor-corrector method to solve fuzzy fractional differential equation.

Max-Min improved Euler method was introduced by Najeeb Alam Khan et al.[25] for solving initial value problem of fuzzy fractional differential equations. Two modified fractional Euler methods used to obtain numerical solutions for linear and nonlinear fuzzy fractional differential equation with triangular and trapezoidal initial values. First method is Max-Min modified fractional Euler method and the second method is Average modified fractional Euler method. The approximate solutions obtained by both methods were compared with exact solutions through tables and graphs[25].

Najeeb Alam Khan et al.[24] compared the numerical simulation of linear and nonlinear fuzzy initial value problems of fractional order with the two method, one of them is Improved Fractional Eulers Method (IFEM), another one is Modified Homotopy Perturbation Method (MHPM). The author elucidated that among the two methods (MHPM, IFEM) IFEM was failed for large values, but proved to be correct for smaller values of dependent variables. On the other hand MHPM gives more accurate approximation for both small and large values of dependent variables. They showed that MHPM is more accurate and efficient than IFEM.

### 3.9 Variational Iteration Method

Several authors [17,18,45] were used iteration procedure to solve fuzzy fractional differential equations. The variational iteration method was applied for solving the fractional differential equations with fuzzy initial condition by Khodadadi et al. [17]. They compared the approximate solutions with their exact solutions to demonstrate the validity and applicability of the proposed method. The authors suggested that the proposed method is efficient, accurate and convenient for solving the fuzzy fractional differential equations with fractional Riemann-Liouville derivatives.

Khodadadi et al. [18] also proposed solutions of fuzzy fractional Riccati differential equations under Caputo H-differentiability by variational iteration method. The solution obtained by the variational iteration method is an infinite power series, which can be expressed in an implicit form with suitable fuzzy initial condition. Here fuzzy number is triangular fuzzy number.

In addition, Zhen-Guo Deng et al. [45] have used fractional variation iteration method to solve fractional differential equations with uncertainty. They adopted the modified Riemann-Liouville derivative.

### 3.10 Predictor-Corrector Method

Recently, Ali Ahmadian et al. [7] have developed fractional predictor-corrector method to solve fuzzy fractional differential equations under the Caputo generalized Hukuhara differentiability. In which, they solved the fuzzy linear fractional relaxation-oscillation problem using fractional predictor-corrector method. In this method the fuzzy fractional differential equation is converted to fuzzy Volterra integral equation and then applied predictor-corrector Method. They have utilized a fractional Adams-Bashforth as predictor and a fractional Adams-Moulton as corrector.

### 3.11 Homotopy Perturbation Method

Smita Tapaswini and Chakraverty [41] proposed homotopy perturbation method (HPM) to solve fuzzy arbitrary order Predator-Prey equations. Fuzziness is appeared in the initial conditions.

### 3.12 Iteration Methods for Hybrid fuzzy fractional differential equations

Rubanraj et al. [32-35] introduced a special case in fuzzy fractional differential equation which is called as Hybrid fuzzy fractional differential equation. They solved this Hybrid fuzzy fractional differential equation with fuzzy initial condition using the following iteration methods:

- (i) Milnes method
- (ii) Adam Bash forth method
- (iii) Runge-Kutta 4th order method for both first and second order differential equations.

The authors compared the solutions obtained by above methods with exact solution through illustrations. Also they provided the definition of the degree of sub element hood of Hybrid fuzzy fractional differential equation. They proved efficiency, accuracy, and validity of proposed methods through illustration.

## 4 Conclusion

This paper comprehend the various method to solve fuzzy fractional differential equations. Since this area is fast growing now, in future, we may apply several other techniques which are being used to solve differential equation involving integer order to fuzzy fractional differential equation.

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