

COMPLEMENTARY VARIATIONAL PRINCIPLES FOR THE BENDING OF A CANTILEVER BY TRANSVERSE FORCE

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Abstract: Complementary Variational Principles is developed for the generalized bending of a cantilever. We have obtained the lower and upper bounds of the functional, a body with arbitrary plan-forms. In particular cases we consider bodies with elliptic and rectangular plan-forms. Results are compared with that of existing ones. This is very interesting basic problems so that further research in experimental work will trim down the gap that exists between the Mathematician and Engineers.

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Introduction

We formulate Complementary Variational Principles for the generalized bending of a finite

Homogenous rectilinearly anisotropic cantilever by transverse force. The line of action of the force is taken through the center of flexure. Therefore, this transverse force produces bending in principle planes not accompanied by twisting (i.e. $\theta = 0$). The direction of the force coincides with the direction of one of the principle axis of inertia of the section. The lateral surface is not loaded, the body forces are absent. One end of the cantilever is rigidly fixed and the other end is acted on by a resultant force P applied in the end plane.

Let x and y axes be taken along the principle axes of inertia of the end section. Let z axis be parallel to the generators with the centroid of the free end section as origin. Here normal stress is taken to be

$$\tau_{33} = \frac{-Pzy}{I} + \tau_{33}(x, y) \quad (1)$$

Where $\frac{-Pzy}{I}$ the normal stress at the cross-section is $z = z_0, 0 \leq z_0 \leq l$ (l is the length of the body) and all the other stresses are independent of z substituting for τ_{33} in the generalized Hooke's law

We obtain

$$\begin{aligned} \frac{\partial u}{\partial x} &= e_{11} = \beta_{11}\tau_{11} + \beta_{12}\tau_{22} + \beta_{13}\tau_{33} + \beta_{14}\tau_{23} + \beta_{15}\tau_{13} + \beta_{16}\tau_{12} + a_{13}D_0 \\ \frac{\partial v}{\partial y} &= e_{22} = \beta_{21}\tau_{21} + \beta_{22}\tau_{22} + \beta_{23}\tau_{33} + \beta_{24}\tau_{23} + \beta_{25}\tau_{13} + \beta_{26}\tau_{12} + a_{23}D_0 \\ \frac{\partial w}{\partial z} &= e_{33} = \left(G_0(x, y) - \frac{pzy}{l} \right) a_{33} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} &= e_{23} = \beta_{14}\tau_{11} + \beta_{24}\tau_{22} + \dots + \beta_{46}\tau_{12} + a_{34}D_0 \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} &= e_{13} = \beta_{15}\tau_{11} + \beta_{25}\tau_{22} + \dots + \beta_{56}\tau_{12} + a_{35}D_0 \\ \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} &= e_{12} = \beta_{16}\tau_{11} + \beta_{26}\tau_{22} + \dots + \beta_{56}\tau_{12} + a_{36}D_0 \end{aligned} \quad (2)$$

Integrating the third equation of (2) w.r.to z , we obtain

$$w = zG_0 - \frac{pa_{33}z^2y}{2I} + W(x, y) \quad (3)$$

Differentiating w in (3) w.r.to x and y and substituting in the 5th and 4th equations of (2),

we obtain expressions for $\frac{\partial v}{\partial z}$ and $\frac{\partial u}{\partial z}$

Which on the integration w.r to z ,

$$v = \frac{pa_{33}z^3}{6I} - \frac{z^2 \left(\frac{\partial G_0}{\partial y} + \frac{a_{34}py}{I} \right)}{2} + z \left(\frac{-\partial w}{\partial y} + \beta_{14}\tau_{11} + \beta_{24}\tau_{22} + \dots + \beta_{46}\tau_{12} + \right.$$

$$\begin{aligned}
v &= \frac{pa_{33}z^3}{6I} - \frac{z^2 \left(\frac{\partial G_0}{\partial y} + \frac{a_{34}py}{I} \right)}{2} + z \left(\frac{-\partial w}{\partial y} + \beta_{14}\tau_{11} + \beta_{24}\tau_{22} + \dots + \beta_{46}\tau_{12} + \right. \\
&\quad \left. a_{34}D_0 / a_{33} \right) V(x, y) \\
u &= -z^2 \left(\frac{\partial G_0}{\partial x} + \frac{a_{35}py}{I} \right) + z \left(\frac{-\partial w}{\partial x} + \beta_{15}\tau_{11} + \beta_{25}\tau_{22} + \dots + \beta_{56}\tau_{13} + a_{35}D_0 / a_{33} \right) U(x, y) \quad (4)
\end{aligned}$$

Differentiating u and v in (4) w.r.to x and y and comparing the resulting expression with first, second and sixth equation of (2), we obtain

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \beta_{11}\tau_{11} + \beta_{12}\tau_{22} + \beta_{13}\tau_{33} + \beta_{14}\tau_{23} + \beta_{15}\tau_{13} + \beta_{16}\tau_{12} + a_{13}D_0 / a_{33} \\
\frac{\partial v}{\partial y} &= \beta_{21}\tau_{21} + \beta_{22}\tau_{22} + \beta_{23}\tau_{33} + \beta_{24}\tau_{23} + \beta_{25}\tau_{13} + \beta_{26}\tau_{12} + a_{23}D_0 / a_{33} \\
\frac{\partial u}{\partial y} &= \beta_{16}\tau_{11} + \dots + a_{36}D_0 / a_{33} \\
\frac{\partial w}{\partial y} &= \beta_{14}\tau_{11} + \dots + a_{34}D_0 / a_{33} + \frac{p(a_{23}y^2 - a_{13}x^2)}{2I - \theta x} \\
\frac{\partial w}{\partial x} &= \beta_{15}\tau_{11} + \beta_{25}\tau_{22} + \dots + a_{35}D_0 / a_{33} + \frac{p(a_{36}y^2 + 2a_{13}xy)}{2I + \theta y} \quad (5)
\end{aligned}$$

The equilibrium equations becomes

$$\begin{aligned}
\tau_{11,1} + \tau_{12,2} &= 0 \\
\tau_{12,1} + \tau_{22,2} &= 0 \\
\tau_{13,1} + \tau_{23,2} &= \frac{py}{I} \quad (6)
\end{aligned}$$

There exist function F and χ satisfying the equilibrium (1.5) such that

$$\begin{aligned}
\tau_{11} &= F_{yy}, \tau_{22} = F_{xx}, \tau_{12} = -F_{xy} \\
\tau_{13} &= \chi_y = \tau_1, \tau_{23} = -\chi_x + \tau_2 \quad (7)
\end{aligned}$$

Now, since

$$\frac{\partial^2}{\partial y^2} \left(\frac{\partial u}{\partial x} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial v}{\partial y} \right) - \frac{\partial^2}{\partial x \partial y} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = 0$$

$$\frac{\partial}{\partial x} \left(\frac{\partial w}{\partial y} \right) - \frac{\partial}{\partial y} \left(\frac{\partial w}{\partial x} \right) = 0 \quad (8)$$

We get, using (7) and (8)

$$\begin{aligned} L_4 F + L_3 \chi &= Q_1 \\ L_3 F + L_2 \chi &= Q_2 \text{ on } \partial B_0 \end{aligned} \quad (9)$$

The conditions on the lateral surface which is not loaded are

$$\begin{aligned} \frac{\partial F}{\partial x} &= C_1, \quad \frac{\partial F}{\partial y} = C_2, \\ \chi &= \int_0^s (\tau_2 dx - \tau_1 dy) + C_3 \text{ On } \partial B_0 \end{aligned} \quad (10a)$$

If we choose $\tau_2 dx - \tau_1 dy$ to be zero on the contour, then we obtain

$$\begin{aligned} \frac{\partial F}{\partial x} &= C_1, \quad \frac{\partial F}{\partial y} = C_2 \\ \chi &= C_3 \text{ on } \partial B_0 \end{aligned} \quad (10b)$$

For simply connected cross section, we get

$$C_1 = 0, C_2 = 0, C_3 = 0 \quad \text{on } \partial B_0 \quad (11)$$

Since on $X_n = Y_n = X = Y = 0$ we get from (10b) that

$$A = 0 = B \quad \text{and} \quad C = \frac{-p}{2Sa_{34}} \quad (12)$$

Where S is cross section area.

Let a section be bounded by the curves

$$y = \pm \sqrt{f(x)} \quad (13)$$

Or

By these curve and by one or two straight lines parallel to the y axis, then by choosing

$$\tau_1 = 0$$

$$\tau_2 = \frac{p(y^2 - f(x))}{l} \quad (14)$$

Then we have

$$\int_0^s (\tau_2 dx - \tau_1 dy) = 0 \quad (15)$$

1.2 COMPLEMENTARY VARIATIONAL PRINCIPLE

However, the use of metric shape functions as trial functions causes the element to exhibit rotational frame dependence as well as interpolation failure under certain conditions (Ooi et al., 2008). By employing bivariate quadratic splines on triangulated quadrangulations, Li and Wang (2006) proposed a new 8-node spline element L8 in 2006. Good results for some problems can be obtained after a relatively complicated mathematical treatment. At the end of last century, Long et al. (1999a, b) established a new natural coordinate system, the quadrilateral area coordinates method, for constructions of the quadrilateral finite element model. On the basis of the new tool, three 8-node models, AQ8-I, AQ8-II and QACM8, were successfully developed by Soh et al. (2000) and Cen et al. (2007), respectively. When all element edges are straight, the interpolation functions for the displacement fields of these elements will possess second-order completeness in both area and Cartesian coordinates, so all the elements can exhibit excellent performance in bending problems, and are insensitive to mesh distortion. However, once any element edge is curved, the above second-order completeness will no longer exist.

Equation (8) with boundary condition (10) can be written in the canonical form as

$$T\varphi = U \quad (16)$$

$$T\varphi = U \quad (17)$$

With

$$\begin{aligned} F = 0, \frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0 \\ \chi = 0 \text{ On } \partial B_0 \end{aligned} \quad (18)$$

Where

$$Q' = [Q_1 \quad Q_2']$$

$$U = [u_1 u_2 u_3 u_4]'$$

$$\varphi = [F \chi]'$$

In (4.16) the operator $T : H(F) \rightarrow H(U)$ is

$$T = \begin{bmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{bmatrix} \quad (19)$$

And $T^* = H(U) \rightarrow H(\varphi)$

$$T^* = \begin{bmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{bmatrix} \quad (20)$$

The operator T and T^* satisfy relationship

$$I(U, T\varphi) = \langle T^*U, \varphi \rangle + \langle \sigma_1 \varphi, U \rangle \partial \beta_0 + \langle \sigma^* U, \varphi \rangle \partial \beta_0 \quad (21)$$

In $\sigma_1 = H(F) \rightarrow H(U)$ is

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \\ \sigma_{31} & \sigma_{32} \\ \sigma_{41} & \sigma_{42} \end{bmatrix} \quad (22)$$

And $\sigma^* = H(u) \rightarrow H(F)$ is

$$\sigma^* = \begin{bmatrix} \sigma_{11}^* & \sigma_{12}^* & \sigma_{13}^* & \sigma_{14}^* \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (23)$$

Now consider the functional

$$I\langle U, \varphi \rangle = \langle U, T\varphi \rangle - \frac{\langle U, U \rangle}{2} - \langle Q', \varphi \rangle - \langle \sigma_1 \varphi, U \rangle \partial \beta_0 + \langle \sigma^* U, \varphi \rangle \sigma B_0 \quad (24)$$

Form (21), we obtain

$$I\langle U, \varphi \rangle = \langle T^*U, \varphi \rangle - \frac{\langle U, U \rangle}{2} - \langle Q', \varphi \rangle \quad (25)$$

Now we construct $J(\varphi_1)$ and $G(U_1)$ in the similar manner as in chapter 1 as follows

$$J(\varphi_1) = I\langle T\varphi_1, T\varphi_1 \rangle = \frac{\langle T\varphi_1, T\varphi_1 \rangle}{2} - \langle Q', \varphi \rangle \quad (26)$$

$$\text{And } G(U_2) = I(U_2, T^{-1}U_2) = -\frac{(U_2, U_2)}{2} \quad (27)$$

Thus the complementary variational principles are

$$G(U_2) \leq G(U) - I(U, \varphi) = J(\varphi) \leq J(\varphi_1) \quad (28)$$

$$V = \frac{1}{2} \int_{B_0} [e_{11}\tau_{11} + e_{22}\tau_{22} + e_{33}\tau_{33} + e_{23}\tau_{23} + e_{13}\tau_{13} + e_{12}\tau_{12}] dx dy \quad (29)$$

$$\begin{aligned} V = & \frac{1}{2} \int_{B_0} [\beta_{11} F_{yy}^2 + 2\beta_{12} F_{yy} F_{xx} - 2\beta_{14} F_{yy} \chi_x + 2\beta_{15} F_{yy} \chi_y - 2\beta_{16} F_{yy} F_{xy} + \beta_{22} F_{xx}^2 - \\ & 2\beta_{24} F_{xx} \chi_x + 2\beta_{25} F_{xx} \chi_y - 2\beta_{26} F_{yy} F_{xx} + \beta_{44} \chi_x^2 - 2\beta_{45} \chi_x \chi_y + \beta_{55} \chi_y^2 - 2\beta_{56} F_{xy} \chi_y + \\ & \beta_{36} F_{xy}^2 + 2\beta_{14} F_{yy} \tau_1 + 2\beta_{15} F_{yy} \tau_1 + \beta_{45} F_{xx} \tau_2 + 2\beta_{25} F_{xx} \tau_1 - 2\beta_{44} \chi_x \tau_2 + 2\beta_{45} \chi_y \tau_1 - \\ & 2\beta_{46} F_{xy} \tau_2 + 2\beta_{55} \chi_y \tau_1 - 2\beta_{56} F_{xy} \tau_1] dx dy + V_1 \end{aligned} \quad (30)$$

Integrating by part the terms involving τ_1 and τ_2 and using the boundary condition (1.10) we get

$$v = \frac{-(T\varphi, T\varphi)}{2} + \langle Q', \varphi \rangle + V_1 \quad (31)$$

Thus the complementary variational principles we obtain

$$G(U_2) \leq -V_2 \leq J(\varphi_1) \quad (32)$$

Since V_1 is independent of F and χ , we get bounded on strain energy by the equation (32)

1.3 STRESS DISTRIBUTION IN A CYLINDER WITH ELLIPTIC PLAN-FORM

In the case in a cylinder with elliptic plan-form, we have

$$f(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$$

The matrix C_{ij} in takes the form

$$\begin{aligned} C_{11} &= \frac{4\{3(\beta_{11}c^2 + c^2\beta_{12} + \beta_{66}) + 3\beta_{22}c^2\}}{a^2} \\ C_{12} &= -\{3\beta_{24} + (\beta_{14} + \beta_{56})c^2\} \\ C_{13} &= 3\beta_{15}c^2 + (\beta_{25} + \beta_{46}) \\ C_{23} &= -2\beta_{45}c^2 \\ C_{22} &= 3\beta_{44}c^2 + \beta_{55} \\ C_{33} &= \beta_{44}c^2 + 3\beta_{55} \end{aligned} \quad (33)$$

Comparing the expressions for C_{ij} from equation (33) and corresponding equation in [7] we find the two are one and the same. Hence the upper bound gives the best solution.

1.4 STRESS DISTRIBUTION IN A CYLINDER WITH RECTANGULAR PLAN-FORM

In the case of a cylinder with rectangular plan-form, we have

$$\begin{aligned} C_{11} &= \frac{256\{(\beta_{22}b^4 + a^4\beta_{11} + 4(\beta_{12} + \beta_{66})/(7a^2b^2))\}}{105} \\ C_{12} &= -32\{(9a^2\beta_{15}/2 + (\beta_{25} + \beta_{46})b^2)\}/105 \\ C_{13} &= 32\{(9b^2\beta_{24}/2 + (\beta_{14} + \beta_{56})a^2)\}/105 \end{aligned}$$

$$C_{23} = -\beta_{45} / 15$$

$$C_{22} = \frac{\beta_{44}}{5} + \frac{a^2 \beta_{55}}{21b^2}$$

$$C_{33} = \frac{\beta_{55}}{5} + \frac{b^2 \beta_{44}}{21a^2} \quad (34)$$

If the plan of the cross section of a cantilever is a plan of elastic symmetry,

We have,

$$a_{14} = a_{24} = a_{34} = a_{46} = 0$$

And

$$a_{15} = a_{25} = a_{35} = a_{56} = 0$$

In (5), (6), (7) and (12), Hence we obtain

$$U = F = V = 0, C = 0$$

$$\tau_{11} = \tau_{22} = \tau_{12} = \tau_{33} = 0$$

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