ON SOME LABELINGS OF LINE GRAPH OF BARBELL GRAPH

P. Agasthi\textsuperscript{1}, N. Parvathi\textsuperscript{2}, K. Thirusangu\textsuperscript{3}

\textsuperscript{1}Department of Mathematics
Tagore Engineering College
Rathinamangalam, Vandalur, Chennai-600 048.
e-mail: agasthipadmanathan@yahoo.com

\textsuperscript{2}Department of Mathematics
SRM University, Kattankulathur-603 203

\textsuperscript{3}Department of Mathematics, S.I.V.E.T College
Gowrivakkam, Chennai-600 073.
e-mail: kthirusangu@gmail.com

Abstract

The graphs considered here are finite, undirected and simple. A Barbell graph $B(p, n)$ is the graph obtained by connecting $n$-copies of a complete graph $k_{p}$ by a bridge. The Line graph whose vertex set is $E(B(p, n))$ and two vertices are adjacent in $L(B(p, n))$ whenever they are incident in $B(p, n)$. In this paper the ways to construct square sum, square difference, Root Mean square, Strongly Multiplicative, Even Mean, Odd Mean, Cordial and Total Cordial labelings for line graph of Barbell graphs are reported.

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\textbf{Key Words and Phrases:} Barbell graph, line graph, square sum labeling, square difference labeling, Root Mean square labeling, Strongly Multiplicative labeling, Even Mean labeling, Odd Mean labeling, Cordial labeling, Total Cordial labeling.
1 Introduction

The graphs considered here are finite, undirected and simple. The concept of graph labeling was introduced by A. Rosa in 1967. Let $G(V,E)$ be a $(p,q)$ graph. The Mean labeling was introduced by S.Somasundram and R.Ponraj [2]. Root square Mean labeling was introduced by S.S.Sandhya, S.Somasundram and S.Anusa in [3]. The square sum and square difference labeling were introduced by Ajiths, Arumugam and Gerimina [4]. The concept of Strongly Multiplicative graphs introduced by Beineke and Hegde [2001]. In this paper we prove that the existence of Square Sum, Square difference, Root Mean Square, Strongly Multiplicative, Even Mean, Odd Mean, Cordial and Total Cordial labeling of line graph of Barbell graph $L(B(p,n))$ for $p \geq 3$ and for all $n \geq 2$.

Definition 1. A Barbell graph $B(p,n)$ is graph obtained by connecting $n$-copies of a complete graph $k_p$ by a bridge.

Definition 2. The Line graph $L(G)$ of a graph $G$ is the graph whose vertex set is $E(G)$ and two vertices are adjacent in $L(G)$ whenever they are incident in $G$.

Definition 3. Let $G$ be a $(p,q)$ graph. A one-one map $f : V(G) \to \{0,1,\ldots,p-1\}$ is said to be a Square Sum labeling if the induced map $f^*(uv) = (f(u))^2 + (f(v))^2$ is injective. It is said to be a Square difference labeling if the induced map $f^*(uv) = (f(u))^2 - (f(v))^2$ is injective.

Definition 4. Let $G$ be a $(p,q)$ graph. A graph $G$ admits Root Mean square labeling if there exist a injective mapping from the vertices of $G$ to set $\{0, 1, \ldots, 2p\}$ such that when each edge $uv$ is assigned the label $f^*(uv) = \sqrt{(f(u))^2 + (f(v))^2}$, then the resulting edge labels are distinct.

Definition 5. A $(p,q)$ graph said to be strongly Multiplicative if there exist a one-one map $f : V(G) \to \{1, 2, \ldots, p\}$ such that the induced map defined by $f^*(uv) = f(u)f(v)$ giving distinct edge values.

Definition 6. A $(p,q)$ graph is called Even Mean graph, if there exist a one-one map $f : V(G) \to \{2, 4, 6, \ldots, 2q\}$ such that
the induced map \( f^*(uv) = \frac{(f(u)+f(v))}{2} \) are distinct.

**Definition 7.** A \((p, q)\) graph is called Odd Mean graph, if there exist a one-one map \( f : V(G) \to \{1, 3, 5, \ldots, (2q - 1)\} \) such that the induced map \( f^*(uv) = \frac{(f(u)+f(v))}{2} \) are distinct.

**Definition 8.** A mapping \( f : V(G) \to \{0, 1\} \) is called binary vertex labeling of \( G \) and \( f(v) \) is called the label of the vertex \( v \) of \( G \) under \( f \).

The induced edge labeling \( f^* : E(G) \to \{0, 1\} \) is given by \( f^*(e = uv) = |f(u) - f(v)| \).

**Definition 9.** A binary vertex labeling of a graph \( G \) is called a cordial labeling if \( |v_f(0) - v_f(1)| \geq 1 \) and \( |e_f(0) - e_f(1)| \leq 1 \). Where \( v_f(0) \) = number of vertices of \( G \) having label 0 under \( f \).

\( v_f(1) \) = number of vertices of \( G \) having label 1 under \( f \).

\( e_f(0) \) = number of edges of \( G \) having label 0 under \( f^* \).

\( e_f(1) \) = number of edges of \( G \) having label 1 under \( f^* \).

A graph \( G \) is called cordial if it admits cordial labeling.

**Definition 10.** A total-cordial labeling of a graph \( G \) with vertex set \( V \) and edge set \( E \) as an cordial labeling if \( |(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))| \leq 1 \).

## 2 Main Results

**Theorem 11.** The Line graph of Barbell graph \( L[(B(p, n))] \) is a square sum graph for \( p \geq 3 \) and for all \( n \geq 2 \).

**Proof.** Let \( G = B(p, n) \) be a graph with \( p \) vertices and \( q \) edges and its line graph \( L[(B(p, n))] \), \( \{v_1, v_2, \ldots, v_n \} \) be the vertices and the edge set be \( \{e_1, e_2, \ldots, e_n\} \).

We define the labeling function \( f : V(G) \to \{0, 1, 2, \ldots, (p - 1)\} \) as follows

\[
\begin{align*}
f(v_i) &= i, \quad 1 \leq i \leq \frac{p(p-1)}{2} \\
f(v_i) &= 0, \quad i = \frac{p(p-1)}{2} + 1 \\
f(v_i) &= i - 1, \quad \frac{p(p-1)}{2} + 2 \leq i \leq \frac{n(p(p-1))}{2} + (n-1)
\end{align*}
\]

Define the induced function on edges as \( f^* : E \to N \) such that \( f^*(v_iv_{i+1}) = (f(v_i))^2 + (f(v_{i+1}))^2 \).
The above defined labeling pattern easy to verify that all the vertex labels are different and we get edge labels are non-zero integers and no edge label is repeated.
Hence \( L[(B(p, n))] \) is a square sum labeling.

**Theorem 12.** The Line graph of Barbell graph \( L[(B(p, n))] \) is a square difference graph for \( p \geq 3 \) and for all \( n \geq 2 \).

**Proof.** Let \( G = B(p, n) \) be a graph with \( p \) vertices and \( q \) edges and it’s line graph \( L[(B(p, n))] \), \( \{v_1, v_2, \ldots, v_{n(p(p-1))}\} \) be the vertices and the edge set be \( \{e_1, e_2, \ldots, e_n\} \).
We define the labeling function \( f: V(G) \rightarrow \{0, 1, 2, \ldots, (p-1)\} \) as follows
\[
\begin{align*}
f(v_i) &= i, \quad 1 \leq i \leq \frac{p(p-1)}{2} \\
f(v_i) &= 0, \quad i = \frac{p(p-1)}{2} + 1 \\
f(v_i) &= i - 1, \quad \frac{p(p-1)}{2} + 2 \leq i \leq \frac{n(p(p-1))}{2} + (n - 1)
\end{align*}
\]
Define the induced function on edges as \( f^*: E \rightarrow N \) such that
\[
f^*(v_iv_{i+1}) = (f(v_i))^2 - (f(v_{i+1}))^2
\]
The above defined labeling pattern easy to verify that all the vertex labels are different and we get edge labels are non-zero integers and no edge label is repeated.
Hence \( L[(B(p, n))] \) is a square difference labeling.

**Theorem 13.** The Line graph of Barbell graph \( L[(B(p, n))] \) is a Root Mean square graph for \( p \geq 3 \) and for all \( n \geq 2 \).

**Proof.** Let \( G = B(p, n) \) be a graph with \( p \) vertices and \( q \) edges and it’s line graph \( L[(B(p, n))] \), \( \{v_1, v_2, \ldots, v_{n(p(p-1))}\} \) be the vertices and the edge set be \( \{e_1, e_2, \ldots, e_n\} \).
We define the labeling function \( f: V(G) \rightarrow \{0, 1, 2, \ldots, 2p\} \) as follows
\[
\begin{align*}
f(v_i) &= i, \quad 1 \leq i \leq \frac{p(p-1)}{2} \\
f(v_i) &= 0, \quad i = \frac{p(p-1)}{2} + 1 \\
f(v_i) &= i - 1, \quad \frac{p(p-1)}{2} + 2 \leq i \leq \frac{n(p(p-1))}{2} + (n - 1)
\end{align*}
\]
Define the induced function on edges as \( f^*: E \rightarrow N \) such that
\[
f^*(v_iv_{i+1}) = \sqrt{(f(v_i))^2 + (f(v_{i+1}))^2}
\]
The above defined labeling pattern easy to verify that all the vertex labels are different and we get edge labels are non-zero integers and no edge label is repeated.

Hence \( L([B(p, n)]) \) is a Root Mean square labeling. 

**Theorem 14.** The Line graph of Barbell graph \( L([B(p, n)]) \) is a Strongly Multiplicative graph for \( p \geq 3 \) and for all even \( n \).

**Proof.** Let \( G = B(p, n) \) be a graph with \( p \) vertices and \( q \) edges and it’s line graph \( L([B(p, n)]) \), \( \{v_1, v_2, \ldots, v_{p(p-1)}\} \) be the vertices and the edge set be \( \{e_1, e_2, \ldots, e_n, e'_1, e'_2, \ldots, e'_{n-1}\} \)

We define the labeling function \( f : V(G) \rightarrow \{1, 2, \ldots, p\} \) as follows

\[
\begin{align*}
    f(v_i) &= \{1, 3, 5, \ldots\}, 1 \leq i \leq \frac{p(p-1)}{2} \\
    f(v_i) &= \{2, 4, 6, \ldots\}, \frac{p(p-1)}{2} + 1 \leq i \leq p(p-1) \\
    f(v_i) &= p(p-1) + f(v_{i-6}), (p(p-1)) + 1 \leq i \leq \frac{n(p(p-1))}{2} \\
    f(v'_i) &= \frac{n(p(p-1))}{2} + i, 1 \leq i \leq n-1
\end{align*}
\]

Define the induced function on edges as \( f^* : E \rightarrow N \) such that

\[
f^*(v_iv_{i+1}) = f(v_i)f(v_{i+1})
\]

The above defined labeling pattern easy to verify that all the vertex labels are different and we get edge labels are non-zero integers and no edge label is repeated.

Hence \( L([B(p, n)]) \) is a Strongly Multiplicative labeling. 

**Theorem 15.** The Line graph of Barbell graph \( L([B(p, n)]) \) is an Odd Mean graph for \( p \geq 3 \) and for all \( n \geq 2 \).

**Proof.** Let \( G = B(p, n) \) be a graph with \( p \) vertices and \( q \) edges and it’s line graph \( L([B(p, n)]) \), \( \{v_1, v_2, \ldots, v_{p(p-1)}\} \) be the vertices and the edge set be \( \{e_1, e_2, \ldots, e_n, e'_1, e'_2, \ldots, e'_{n-1}\} \)

We define the labeling function \( f : V(G) \rightarrow \{1, 3, 5, \ldots, (2q - 1)\} \) as follows

\[
\begin{align*}
    f(v_i) &= \{1, 3, 5, \ldots\}, 1 \leq i \leq \frac{p(p-1)}{2} \\
    f(v_i) &= p(p-1) + f(v_{i-6}), \frac{p(p-1)}{2} + 1 \leq i \leq \frac{n(p(p-1))}{2} \\
    f(v'_i) &= n(p(p-1)) + (i), i = 1 \\
    f(v'_i) &= p(p-1) + f(v'_{i-1}), 2 \leq i \leq n - 1
\end{align*}
\]
Define the induced function on edges as $f^* : E \to N$ such that

$$f^*(u_iu_{i+1}) = \frac{f(u_i) + f(u_{i+1})}{2}$$

The above defined labeling pattern easy to verify that all the vertex labels are different and we get edge labels are non-zero integers and no edge label is repeated.

Hence $L[(B(p, n))]$ is an Odd Mean labeling.

**Theorem 16.** The Line graph of Barbell graph $L[(B(p, n))]$ is an Even Mean graph for $p \geq 3$ and for all $n \geq 2$.

**Proof.** Let $G = B(p, n)$ be a graph with $p$ vertices and $q$ edges and it’s line graph $L[(B(p, n))]$, $\{v_1, v_2, \ldots, v_{n(p(p-1)/2)}\}$ be the vertices and the edge set be $\{e_1, e_2, \ldots, e_n, e'_1, e'_2, \ldots, e'_{n-1}\}$

We define the labeling function $f : V(G) \to \{2, 4, 6, \ldots, 2q\}$ as follows

- $f(v_i) = \{2, 4, 6, \ldots\}, 1 \leq i \leq p(p-1)/2$
- $f(v_i) = p(p-1) + f(v_{i-6}), \frac{p(p-1)}{2} + 1 \leq i \leq \frac{n(p(p-1))}{2}$
- $f(v'_i) = n(p(p-1)) + (i+1), i = 1$
- $f(v''_i) = p(p-1) + f(v'_{i-1}), 2 \leq i \leq n - 1$

Define the induced function on edges as $f^* : E \to N$ such that

$$f^*(v_iu_{i+1}) = \frac{f(v_i) + f(u_{i+1})}{2}$$

The above defined labeling pattern easy to verify that all the vertex labels are different and we get edge labels are non-zero integers and no edge label is repeated.

Hence $L[(B(p, n))]$ is an Even Mean labeling.

**Theorem 17.** The Line graph of Barbell graph $L[(B(p, n))]$ is a Cordial graph for $p \geq 3$ and for all $n \geq 2$.

**Proof.** Let $G = B(p, n)$ be a graph with $p$ vertices and $q$ edges and it’s line graph $L[(B(p, n))]$, $\{v_1, v_2, \ldots, v_{n(p(p-1)/2)}\}$ be the vertices and the edge set be $\{e_1, e_2, \ldots, e_n, e'_1, e'_2, \ldots, e'_{n-1}\}$

We define the labeling function $f : V(G) \to \{0, 1\}$ by

The vertices extracted from the odd copies of the edges by $\{1, 0, 0\}$ and the even copies of the edges by $\{1, 1, 1\}$ and we label the intermediated vertices by
\[ f(v'_i) = \begin{cases} 
0, & \text{for } n \equiv 0 \pmod{2} \\
1 & \leq i \leq n - 1
\end{cases} \]
\[ f(v'_i) = \begin{cases} 
1, & \text{for } n \equiv 1 \pmod{2}, i = 1 \\
0, & \text{for } n \equiv 1 \pmod{2}, 2 \leq i \leq n - 1
\end{cases} \]

Thus the entire \((4n - 1)\) vertices are labeled in such a way that the number of vertices labeled with ‘0’ are \((2n - 1)\) and the number of vertices labeled with ‘1’ are \(2n\).

Define the induced function on edges as \(f^* : E \to \{0, 1\}\) is defined such that
\[ f^*(v_i v_{i+1}) = |f(v_i) - f(v_{i+1})| \]
\[ f^*(v_i v'_i) = |f(v_i) - f(v'_i)| \]
\[ f^*(v'_i v'_{i+1}) = |f(v'_i) - f(v'_{i+1})| \]

Using the induced function, we see that \((np + n - 3)\) edges receive label ‘0’ and ‘1’. Thus the entire \(2(np + n - 3)\) edges are labeled in such a way that the number of edges labeled ‘1’ and the number of edges labeled ‘0’ are same as \((np + n - 3)\). Thus in each cases we have \(|v_f(0) - v_f(1)| \leq 1\) and \(|e_f(0) - e_f(1)| \leq 1\).

Hence \(L[(B(p, n))]\) is Cordial.

**Theorem 18.** The Line graph of Barbell graph \(L[(B(p, n))]\) is a Total Cordial graph for \(p = 3\) and for all \(n \geq 2\).

**Proof.** Let \(G = B(p, n)\) be a graph with \(p\) vertices and \(q\) edges and it’s line graph \(L[(B(p, n))], \{v_1, v_2, \ldots, v_p, v'_1, v'_2, \ldots, v'_{n-1}\}\) be the vertices and the edge set be \(\{e_1, e_2, \ldots, e_n, e'_1, e'_2, \ldots, e'_{n-1}\}\)

We define the labeling function \(f : V(G) \to \{0, 1\}\)

The vertices extracted from the odd copies of the edges by \(\{1, 0, 0\}\) and the even copies of the edges by \(\{1, 1, 1\}\) and we label the intermediated vertices by
\[ f(v'_i) = \begin{cases} 
0, & \text{for } n \equiv 0 \pmod{2} \\
1 & \leq i \leq n - 1
\end{cases} \]
\[ f(v'_i) = \begin{cases} 
1, & \text{for } n \equiv 1 \pmod{2}, i = 1 \\
0, & \text{for } n \equiv 1 \pmod{2}, 2 \leq i \leq n - 1
\end{cases} \]

Thus the entire \((4n - 1)\) vertices are labeled in such a way that the number of vertices labeled with ‘0’ are \((2n - 1)\) and the number of vertices labeled with ‘1’ are \(2n\).

Define the induced function on edges as \(f^* : E \to \{0, 1\}\) is
defined such that
\[ f^*(v_i v_{i+1}) = |f(v_i) - f(v_{i+1})| \]
\[ f^*(v_i v'_i) = |f(v_i) - f(v'_i)| \]
\[ f^*(v'_i v'_{i+1}) = |f(v'_i) - f(v'_{i+1})| \]

Using the induced function, we see that \((np+n-3)\) edges receive label ‘0’ and ‘1’. Thus the entire \(2(np+n-3)\) edges are labeled in such a way that the number of edges labeled ‘1’ and the number of edges labeled ‘0’ are same as \((np+n-3)\). Thus in each cases we have
\[ |(v_f(0)+e_f(0)) - (v_f(1)+e_f(1))| \leq 1. \]

Hence \(L[(B(p, n))]\) is Total Cordial.

\[ \square \]

3 Conclusion

In this paper we have examined the existence of square sum, square difference, Root Mean square, Strongly Multiplicative, Odd mean, Even mean , Cordial, Total Cordial labeling for line graph of Barbell graph \(L[(B(p, n))]\) for all \(p \geq 3\) and for all \(n \geq 2\). Further investigation can be done to obtain the above labeling for some class of graphs.

References


