A study on one edge isolated domination in graphs

V. Mohana Selvi\textsuperscript{1} R. Udhayakumari\textsuperscript{2} M. Durka Devi\textsuperscript{3}
\textsuperscript{1}Assistant professor, Department of mathematics, Nehru Memorial College, Puthanampatti. vmohanaselvi@gmail.com
\textsuperscript{2}Assistant Professor of Mathematics, Dhanalakshmi Srinivasan College of Arts and Science, Perambalur-621212. udhaya.maths@gmail.com
\textsuperscript{3}M.Phil Research Scholar in Mathematics, Nehru Memorial College, Puthanampatti, Tiruchirapalli-621 007. durkadevims@gmail.com

Abstract

In this paper one edge isolated domination is introduced for simple graph by restricting one isolated edge in dominating set of the graphs. This domination parameter is studied for some standard graphs and obtained bounds in terms of elements of G.

Key Words and Phrases: domination number, edge domination number, independent domination number, independent edge domination number, one edge isolated domination number.

AMS Subject Classification: 05C69.

1 Introduction

In this paper \( G = (V, E) \) a finite, simple, connected and undirected graph has \( p \)-vertices and \( q \)-edges. Terms not defined here are used in the sense of Harary [2]. A set \( D \) of vertices of a graph \( G = (V, E) \) is a dominating set of \( G \) if every vertex in \( V \setminus D \) is adjacent to some vertex in \( D \). The domination number \( \gamma(G) \) of \( G \) is the minimum cardinality of all dominating sets of \( G \). This concept was introduced by Ore in [1]. A dominating set \( D \) of a graph \( G = (V, E) \) is an independent dominating set if the induced subgraph \( \langle D \rangle \) has no edges. The independent
domination number $\gamma_i(G)$ is the minimum cardinality of independent dominating sets of $G$. A set $X$ of edges in a graph $G = (V, E)$ is called an edge dominating set of $G$ if every edge in $E - F$ is adjacent to at least one edge in $X$. The edge domination number $\gamma'(G)$ of a graph $G$ is the minimum cardinality of an edge dominating sets of $G$. An edge dominating set $X$ of a graph $G = (V, E)$ is an independent edge dominating set if the induced subgraph $\langle X \rangle$ has only isolated edges. The independent edge domination number $\gamma'_i(G)$ is the minimum cardinality of independent edge dominating sets of $G$.

2 One edge isolated domination number

Definition 1. A dominating set $S \subseteq V$ of $G$ is said to be an one edge isolated dominating set (EID-set) of $G$ if the induced subgraph $\langle S \rangle$ has one isolated edge. The minimum cardinality of the EID-sets of $G$ is called the one edge isolated domination number of $G$ and is denoted by $\gamma_{EI}(G)$. Also an EID-set with minimum cardinality $\gamma_{EI}(G)$ is denoted by $\gamma_{EI}$-set of $G$.

Example

![Graph G](image.png)

Figure 1

For the graph $G$ in figure 1, the vertex set $S = \{v_2, v_4, v_6, v_8\}$ is a $\gamma_{EI}$-set since the induced subgraph $\langle S \rangle$ has an isolated edge and hence $\gamma_{EI}(G) = 4$.

3 The exact values of one edge isolated domination number $\gamma_{EI}(G)$ for some standard graphs.

Theorem 2. For the complete graph $K_n$, $\gamma_{EI}(K_n) = 2, n \geq 2$.

Proof. Let $G$ be the complete graph $K_n$ with at least two vertices and $V(G) = \{v_1, v_2, ..., v_n\}$ be the vertex set of $G$. Then the set $S = \{v_1, v_2\}$ form an EID set of $G$.

Hence

$$\gamma_{EI} \leq |S| = 2$$

(1)

Suppose $S$ is a $\gamma_{EI}$ set of $G$ then the induced subgraph $\langle S \rangle$ must contain one isolated edge gives $|S| \geq 2$.

Hence

$$\gamma_{EI} = |S| \geq 2.$$  

(2)
Then the result follows from equations (1) and (2).

**Theorem 3.** For the complete bipartite graph $K_{m,n}$, $\gamma_{EI}(K_{m,n}) = 2$, $m \geq 2, n \geq 2$.

**Proof.** Let $G$ be a complete bipartite graph $K_{m,n}$ with at least 4 vertices. Let $V(G) = \{u_1, u_2, \ldots, u_m\} \cup \{v_1, v_2, \ldots, v_n\}$, then the set $S = \{u_1, v_1\}$ form an EID set of $G$. Hence

$$\gamma_{EI} \leq |S| = 2.$$  \hfill (3)

Suppose $S$ is a $\gamma_{EI}$ set of $G$ then the induced subgraph $\langle S \rangle$ must contain at least one isolated edge gives $|S| \geq 2$.

Hence

$$\gamma_{EI} = |S| \geq 2.$$  \hfill (4)

Then the result follows from equations (3) and (4).

**Theorem 4.** For the cycle graph $C_n$, $\gamma_{EI}(C_n) = \lceil \frac{(n+2)}{3} \rceil$, $n \geq 3$.

**Proof.** Let $G$ be a cycle graph $C_n$ with at least 3 vertices and $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of $G$. Then the set $S=\{v_1, v_2\} \cup \{v_{3i+2}\}, i = 1, 2, \ldots, \lceil \frac{(n-2)}{3} \rceil$ form an EID set of $G$. Therefore, $|S| = 2 + \lceil \frac{(n+2)}{3} \rceil$. Hence,

$$\gamma_{EI} \leq |S| = \frac{(n+2)}{3}.$$  \hfill (5)

Let $S$ be a $\gamma_{EI}$ set of $G$ then $S$ must contain at least one isolated edge and the domination property gives $S$ has at least $2 + \lceil \frac{(n-2)}{3} \rceil$ vertices. Hence

$$|S| \geq 2 + \frac{(n-2)}{3}.$$  \hfill (6)

Then the result is proved from equations (5) and (6).

**Theorem 5.** For the wheel graph $W_n$, $\gamma_{EI}(W_n) = 2$, $n \geq 4$.

**Proof.** Let $G$ be a wheel graph $W_n$ with at least 4 vertices. Let the vertex set of $G$ be $V(G) = \{u, v_1, v_2, \ldots, v_{n-1}\}$. Then the set $S = \{u, v_1\}$ form an EID set of $G$. Therefore $|S| = 2$. Hence

$$\gamma_{EI} \leq |S| = 2.$$  \hfill (7)

Let $S$ be a $\gamma_{EI}$ set of $G$ then $S$ must contain at least one isolated edge and dominating set of $G$ gives $S$ has at least 2 vertices. Hence,

$$\gamma_{EI} = |S| \geq 2.$$  \hfill (8)

From equations (7) and (8) we get $\gamma_{EI} = 2$. 

\textbf{Theorem 6.} For the star graph $K_{1,n}$, $\gamma_{EI}(K_{1,n}) = 2$, $n \geq 2$.

\textit{Proof.} Let $G$ be a star graph $K_{1,n}$ with at least 2 vertices. Let the vertex set of $G$ be $V(G) = \{u, v_1, v_2, \ldots, v_n\}$. Then the set $S = \{u, v_1\}$ form an EID set of $G$. Therefore $|S| = 2$. Hence,
\begin{equation}
\gamma_{EI} \leq |S| = 2. \tag{9}
\end{equation}
Let $S$ be a $\gamma_{EI}$ set of $G$ then $S$ must contain at least one isolated edge and dominating set of $G$ gives $S$ has at least 2 vertices. Hence
\begin{equation}
\gamma_{EI} = |S| \geq 2. \tag{10}
\end{equation}
Hence the result follows from equations (9) and (10).

\textbf{Theorem 7.} For the fan graph $F_{1,n}$, $\gamma_{EI}(F_{1,n}) = 2$, $n \geq 3$.

\textit{Proof.} Let $G$ be a fan graph $F_{1,n}$ with at least 3 vertices. Let the vertex set of $G$ be $V(G) = \{u, v_1, v_2, \ldots, v_n\}$. Then the set $S = \{u, v_1\}$ form an EID set of $G$. Therefore, $|S| = 2$. Hence
\begin{equation}
\gamma_{EI} \leq |S| = 2. \tag{11}
\end{equation}
Let $S$ be a $\gamma_{EI}$ set of $G$ then $S$ must contain at least one isolated edge and dominating set of $G$ gives $S$ has at least 2 vertices. Hence,
\begin{equation}
\gamma_{EI} = |S| \geq 2. \tag{12}
\end{equation}
Equations (11) and (12) proves the result.

\textbf{Theorem 8.} For the friendship graph $C_{m}^{3}$, $\gamma_{EI}(C_{m}^{3}) = 2$, $m \geq 1$.

\textit{Proof.} Let $G$ be a friendship graph $C_{m}^{3}$ with at least 3 vertices. Let the vertex set of $G$ be $V(G) = \{u, v_1, v_2, \ldots, v_{n-1}\}$. Then the set $S = \{u, v_1\}$ form an EID set of $G$. Therefore, $|S| = 2$. Hence,
\begin{equation}
\gamma_{EI} \leq |S| = 2. \tag{13}
\end{equation}
Let $S$ be a $\gamma_{EI}$ set of $G$ then $S$ must contain at least one isolated edge and dominating set of $G$ gives $S$ has at least 2 vertices. Hence
\begin{equation}
\gamma_{EI} = |S| \geq 2. \tag{14}
\end{equation}
The result is followed from equations (13) and (14).

\textbf{Theorem 9.} For bistar tree $B_{n,n}$, $\gamma_{EI}(B_{n,n}) = 2$, $n \geq 2$.

\textit{Proof.} Let $G$ be a bistar tree $B_{n,n}$ with at least 6 vertices. Let the vertex set of $G$ be $V(G) = \{u, v, u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$. Then the set $S = \{u, v\}$ form an EID set of $G$. Therefore $|S| = 2$. Hence
\begin{equation}
\gamma_{EI} \leq |S| = 2. \tag{15}
\end{equation}
Let $S$ be a $\gamma_{EI}$ set of $G$ then $S$ must contain at least one isolated edge and dominating set of $G$ gives $S$ has at least 2 vertices. Hence,

$$\gamma_{EI} = |S| \geq 2.$$  \hfill (16)

Equations (15) and (16) prove the result.

**Theorem 10.** For the crown graph $C_n^+$, $\gamma_{EI}(C_n^+) = n$, for $n \geq 3$.

**Proof.** Let $G$ be a crown graph $C_n^+$ with at least 6 vertices and $V(G) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ be a vertex set of $G$. The vertex set $S = \{u_1, u_2\} \cup \{v_3, v_4, \ldots, v_n\}$ is an one edge isolated dominating set of $G$. Hence

$$\gamma_{EI}(C_n^+) \leq |S| = 2 + n - 2 = n.$$  \hfill (17)

Let $S$ be the $\gamma_{EI}$-set of $G$ then $S$ must contain at least one isolated edge and dominating set of $G$ gives $S$ has at least 2 vertices. Hence

$$\gamma_{EI}(C_n^+) = |S| \geq n$$  \hfill (18)

Then, the result follows from (17) and (18).

**Theorem 11.** For the graph $K_n^+$, $\gamma_{EI}(K_n^+) = n$, for $n \geq 2$.

**Proof.** Let $G$ be a $K_n^+$ graph with at least 4 vertices and $V(G) = \{v_1, v_2, \ldots, v_n\}$ be a vertex set of $G$. The vertex set $S = \{v_1, v_2\}$ is an EID-set of $G$. Hence

$$\gamma_{EI}(G) \leq |S| = n.$$  \hfill (19)

Let $S$ be the $\gamma_{EI}$-set of $G$. since $S$ must contain at least one isolated edge and dominating set of $G$ gives $S$ has at least 2 vertices. Hence

$$\gamma_{EI}(G) = |S| \geq n.$$  \hfill (20)

From (19) and (20), $\gamma_{EI}(G) = n$.

**Theorem 12.** For Corona graph $P_n^+$, $\gamma_{EI}(P_n^+) = n$, for $n \geq 2$.

**Proof.** Let $G$ be a corona graph with at least 4 vertices and $V(G) = \{v_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ be a vertex set of $G$. The vertex set $S = \{u_1, u_2\} \cup \{v_3, v_4, \ldots, v_n\}$ is an one edge isolated dominating set of $G$. Hence

$$\gamma_{EI}(G) \leq |S| = 2 + n - 2.$$  \hfill (21)

Let $S$ be the $\gamma_{EI}$-set of $G$ then $S$ must contain at least one isolated edge and dominating set of $G$ gives $S$ has at least 2 vertices. Hence

$$\gamma_{EI}(G) = |S| \geq n.$$  \hfill (22)

Then, the result follows from (21) and (22).
**Theorem 13.** For the prism graph $C_3 \times P_n$, $\gamma_{EI}(C_3 \times P_n) = n$, for $n \geq 2$.

**Proof.** Let $G$ be a prism graph with at least 6 vertices and $V(G) = \{v_1, v_2, v_3, \ldots, v_{3n}\}$ be a vertex set of $G$. The vertex set $S = \{v_1, v_4\} \cup \{v_9, v_{12}, \ldots, v_{3n}\}$ is an one edge isolated dominating set of $G$. Hence

$$\gamma_{EI}(G) \leq |S| = 2 + n - 2 = n$$

Let $S$ be the $\gamma_{EI}$-set of $G$ then $S$ must contain at least one isolated edge and dominating set of $G$ gives $S$ has at least 2 vertices. Hence

$$\gamma_{EI}(G) = |S| \geq n.$$  \hspace{1cm} (24)

Then the result follows from (23) and (24).

\[\square\]

## 4 Bounds

**Theorem 14.** For any graph $G$, $\gamma(G) \leq \gamma_i(G) \leq \gamma_{EI}(G)$.

**Proof.** Since every EID-set is an ID-set of $G$ and every ID-set set of $G$ is a dominating set of $G$ gives $\gamma_i(G) \leq \gamma_{EI}(G)$ and $\gamma(G) \leq \gamma_i(G)$. \hspace{1cm} \[\square\]

**Theorem 15.** For any $(p, q)$ graph $G$, $2 \leq \gamma_{EI}(G) \leq \lceil \frac{p}{2} \rceil$, $p \geq 3$.

**Proof.** Let $G = (V, E)$ be a graph with at least 3 vertices and $S$ is an EID-set of $G$. Since every EID-set must contain one isolated edge gives $|S| \geq 2$ and hence the lower bound. For the upper bound, suppose $\gamma_{EI}(G) > \lceil \frac{p}{2} \rceil$ then $S$ is either not contain an isolated edge or $S$ is not minimum EID-set of $G$ and hence the result. \hspace{1cm} \[\square\]

**Theorem 16.** For any graph $G$, $\gamma(G) = \gamma_{EI}(G)$, if $G$ contains the set of end vertices.

**Proof.** Let $G$ be a graph with set of end vertices. Then the order $p$ is even. All the isolated vertices of $G$ forms the $\gamma$-set and hence

$$\gamma(G) = \frac{p}{2}. \hspace{1cm} (25)$$

All the isolated vertices except two vertices, take one isolated edge forms $\gamma_{EI}$-set and hence,

$$\gamma_{EI}(G) = \frac{p}{2} - 2 + 2 = \frac{p}{2}. \hspace{1cm} (26)$$

Then the result follows from (25) and (26). \hspace{1cm} \[\square\]

**Theorem 17.** For any graph $G$, $\gamma(G) + \gamma_{EI}(G) \leq p$, $p \geq 3$ and the bound is sharp for the graphs, which are contains the set of end vertices.

**Proof.** Let $G$ be a graph with at least 3 vertices. Since $\gamma(G) \leq \lceil \frac{p}{2} \rceil$ and $\gamma_{EI}(G) \leq \lceil \frac{p}{2} \rceil$ gives the result. The sharp bound is followed from theorem 16. \hspace{1cm} \[\square\]
5 Conclusion

In this paper we introduced One edge isolated domination number for some standard graphs like Complete graph, Complete bipartite graph, Cycle, Wheel graph, Star graph, Fan graph, Friendship graph, Bistar tree, Crown graph $C_n^+$, Corona graph $K_n^+$, Corona graph $P_n^+$, Prism graph. Also found its bounds and studied the relationship with other domination parameters.

References


