S-DERIVATION PAIRS AND JORDAN S-DERIVATION PAIRS ON SEMIRING

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Abstract: Based on the works on derivations on rings and near rings, in 2010, we introduced the notion of derivation on semirings. In our earlier paper [9], we introduced the notion of Derivation Pairs and Jordan Derivation Pairs on semirings and prove some results. In this paper, we introduce the notion of S-Derivation Pairs and Jordan S-Derivation Pairs on a semiring and analyze their properties.

Key Words: derivation on Semirings, Jordan derivation on semirings, derivation pairs and Jordan derivation pairs on semirings, S-derivation pairs and Jordan S-derivation pairs on semirings

1. Introduction

The notion of derivation pair and Jordan derivation pair on a rings were introduced by Masjeed and Altay [7]. In [1], the author have discussed the notion of derivation pairs and Jordan derivation pairs on Γ-ring.

Motivated by this, in our earlier work, we introduced the notion of Derivation Pairs and Jordan Derivation Pairs on semirings and studied its properties. Now, we introduce the notion of S-Derivation Pairs and Jordan S-Derivation Pairs on Semirings and prove some elegant results.
2. Preliminaries

**Definition 2.1.** [3] A semiring \((S, +, \cdot)\) is a non-empty set \(S\) together with two associative binary operations, + and \(\cdot\), such that the two distributive laws are satisfied. That is, a semiring \((S, +, \cdot)\) is a non-empty set \(S\) together with two binary operations, + and \(\cdot\), such that

1. \((S, +)\) is a semigroup.
2. \((S, \cdot)\) is a semigroup.
3. For all \(a, b, c \in S\), \(a \cdot (b + c) = a \cdot b + a \cdot c\) and \((b + c) \cdot a = b \cdot a + c \cdot a\).

A semiring \((S, +, \cdot)\) is said to be additively commutative if \((S, +)\) is a commutative semigroup.

**Definition 2.2.** [3] Let \((S, +, \cdot)\) be a semiring. An element \(\alpha\) of \(S\) is called additively left cancellative if for all \(\beta, \gamma \in S\), \(\alpha + \beta = \alpha + \gamma \Rightarrow \beta = \gamma\)

If every element of the semiring \(S\) is additively left cancellative, it is called an additively left cancellative semiring.

\(\alpha \in S\) is called additively right cancellative if for all \(\beta, \gamma \in S\), \(\beta + \alpha = \gamma + \alpha \Rightarrow \beta = \gamma\)

If every element of the semiring \(S\) is additively right cancellative, it is called an additively right cancellative semiring.

**Definition 2.3.** [3] A Semiring \((S, +, \cdot)\) is additively cancellative, if it is both additively left and right cancellative semiring.

**Definition 2.4.** [5] Let \((S, +, \cdot)\) be a semiring. An additive mapping \(D : S \to S\) is called a derivation on \(S\) if \(D(xy) = D(x)y + xD(y), \forall x, y \in S\).

**Definition 2.5.** [8] Let \((S, +, \cdot)\) be a semiring. An additive mapping \(d : S \to S\) is called a Jordan derivation on a semiring \(S\) if \(d(x^2) = d(x)x + xd(x) \forall x \in S\).

**Definition 2.6.** [8] Let \(S\) be a semiring. A left \(S\)-semimodule is a commutative monoid \((M, +)\) with additive identity \(0_M\) for which we have a function \(S \times M \to M\), denoted by \((s, m) \mapsto sm\) and called scalar multiplication, which satisfies the following conditions. For all elements \(s\) and \(s'\) of \(S\) and all elements \(m\) and \(m'\) of \(M\):

1. \((ss')m = s(s'm)\)
2. \(s(m + m') = sm + sm'\)
3. \((s + s')m = sm + s'm\)

4. \(1_S m = m\)

If \(V(M) = M\), then \(M\) is an \(S\)-module where \(V(M)\) is the set of all elements of \(M\) having additive inverse.

**Definition 2.7.** [9] Let \(S\) be a Semiring and \(X\) be a \(S\)-module. The additive maps \(d, g : S \rightarrow X\) are called *derivation pair*, denoted by \((d, g)\) if they satisfy the following conditions:

\[
d(xy) = d(x)y + xg(y) \quad \forall x, y \in S
\]

\[
g(xy) = g(x)y + xd(y) \quad \forall x, y \in S
\]

The Pair \((d, g)\) is called a *Jordan derivation pair* if:

\[
d(x^2) = d(x)x + xg(x) \quad \forall x \in S
\]

\[
g(x^2) = g(x)x + xd(x) \quad \forall x \in S
\]

### 3. S-Derivation Pairs and Jordan S-Derivation Pairs

**Definition 3.1.** Let \(S\) be a Semiring and \(X\) be a \(S\)-module. Let \(d, g : S \rightarrow X\) be additive mappings. The pair \((d, g)\) is called *\(S\)-derivation pair* if satisfies the following equations

\[
d(xy) = d(x)y + xg(y) \quad \forall x, y \in S
\]

\[
g(xy) = g(x)y + xd(y) \quad \forall x, y \in S
\]

The pair \((d, g)\) is called *Jordan \(S\)-derivation pair* if

\[
d(x^2) = d(x)x + xg(x) \quad \forall x \in S
\]

\[
g(x^2) = g(x)x + xd(x) \quad \forall x \in S
\]

**Example 3.2.** Let \(S\) be a non commutative semiring and \(X\) be a \(S\)-module. Let \(a, b \in S\) such that \(xa = xb = 0, \forall x \in S\). Define \(d, g : S \rightarrow X\) as follows \(d(x) = ax, g(x) = bx\). Then \((d, g)\) is both a \(S\)-derivation pair and a Jordan \(S\)-derivation pair on \(S\).

**Remark 3.3.** Every \(S\)-derivation pair is a Jordan \(S\)-derivation pair. But converse is not true.
Example 3.4. Let $S$ be a 2-torsion free non commutative semiring and $X$ a $S$–module.

Let $a \in S$ such that $xax = 0$, $\forall x \in S$, but $xay \neq 0$ for some $y \neq x, y \in S$. Define an additive pair $d, g : S \rightarrow X$ as follows $d(x) = xa + ax, g(x) = xa + ax$.

Then $(d, g)$ is a Jordan $S$–derivation pair but not a $S$–derivation pair.

Remark 3.5. Every $S$–derivation pair is a derivation pair. But converse is not true, in general.

Example 3.6. Consider the semiring $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbb{N} \cup \{0\} \right\}$

and the $R$–module $X = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbb{Z} \right\}$

Let $d, g : R \rightarrow X$ be additive mappings on $R$ defined as follows:

\[ d \left( \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 2b \\ 0 & 0 \end{pmatrix} \]

and

\[ g \left( \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \]

Then we can easily check that $(d, g)$ is a derivation pair but not a $S$–derivation pair.

Definition 3.7. Let $S$ be an additively commutative semiring and $X$ a $R$–module. Let $(d_1, g_1)$ and $(d_2, g_2)$ be $S$–derivation pairs on $R$. Then $d_1, d_2, g_1, g_2$ are all derivations on $S$ with values in $X$. Let $d = d_1 + d_2$ and $g = g_1 + g_2$. Then both $d$ and $g$ are additive mappings on $S$. We define the pair $(d, g)$ as $(d, g) = (d_1 + d_2, g_1 + g_2)$ called the sum of two $S$–derivation pairs.

Similarly, we can define the sum of two Jordan $S$–derivation pairs.

Lemma 3.8. Let $S$ be an additively commutative semiring and $X$ a $S$–module. Then

1. Sum of two $S$–derivation pairs is again a $S$–derivation pair.
2. Sum of two Jordan $S$–derivation pairs is again a Jordan $S$–derivation pair.
**Proof.**

1. Let \((d_1, g_1)\) and \((d_2, g_2)\) be \(S\)–derivation pairs on \(S\).

\[
d(xy) = (d_1 + d_2)(xy) = d_1(xy) + d_2(xy) = d_1(x)y + xg_1(y) + d_2(x)y + xg_2(y) = (d_1 + d_2)(x)y + x(g_1 + g_2)(y), \forall x, y \in S \tag{1}
\]

\[
g(xy) = (g_1 + g_2)(xy) = g_1(x)y + xd_1(y) + g_2(x)y + xd_2(y) = (g_1 + g_2)(x)y + x(d_1 + d_2)(y), \forall x, y \in S. \tag{2}
\]

From (1) and (2), we have \((d_1 + d_2, g_1 + g_2)\) is a \(S\)–derivation pair on \(S\).

2. Let \((d_1, g_1)\) and \((d_2, g_2)\) be Jordan \(S\)–derivation pairs on \(S\).

\[
d(x^2) = (d_1 + d_2)(x^2) = d_1(x^2) + d_2(x^2) = d_1(x)x + xg_1(x) + d_2(x)x + xg_2(x) = (d_1 + d_2)(x)x + x(g_1 + g_2)(x), \forall x \in S, \tag{3}
\]

\[
g(x^2) = (g_1 + g_2)(x^2) = g_1(x)x + xd_1(x) + g_2(x)x + xd_2(x) = (g_1 + g_2)(x)x + x(d_1 + d_2)(x), \forall x \in S. \tag{4}
\]

From (3) and (4), we have \((d_1 + d_2, g_1 + g_2)\) is a Jordan \(S\)–derivation pair on \(S\).

**Lemma 3.9.** Let \(S\) be a additively commutative semiring and \(X\) a \(S\)–module. Let \((d, g)\) be a \(S\)–derivation pair on \(S\). Then \(d + g\) is a derivation.

**Proof.** Since \((d, g)\) is a \(S\)–derivation pair on \(S\), we have

\[
d(xy) = d(x)y + xg(y) \quad \forall x, y \in S, \tag{1}
\]

\[
g(xy) = g(x)y + xd(y) \quad \forall x, y \in S. \tag{2}
\]

Adding (1) and (2), we get \((d + g)(xy) = (d + g)(x)y + x(d + g)(y) \quad \forall x, y \in S\)

Hence \(d + g\) is a derivation on \(S\).

**Lemma 3.10.** Let \(S\) be a additively commutative semiring and \(X\) a \(S\)–module. Let \((d, g)\) is a Jordan \(S\)–derivation pair on \(S\). Then \(d + g\) is a Jordan derivation as well as a derivation on \(S\).
Proof. Since \((d, g)\) is a Jordan \(S\)-derivation pair, we have
\[
d(x^2) = d(x)x + xg(x) \quad \forall \ x \in S
\] (1)
and
\[
g(x^2) = g(x)x + xd(x) \quad \forall \ x \in S
\] (2)

Adding (1) and (2), we get
\[
(d + g)(x^2) = (d + g)(x)x + x(d + g)(x) \quad \forall \ x \in S,
\] (3)

implying \(d + g\) is a Jordan derivation on \(S\). Linearization of (3), gives
\[
(d + g)(xy + yx) = (d + g)(x)y + x(d + g)(y) + (d + g)(y)x + y(d + g)(x)
\]
\[
\quad 2(d + g)(xy) = 2[(d + g)(x)y + x(d + g)(y)] \quad \text{[because } S \text{ is commutative}]
\]
\[
(d + g)(xy) = (d + g)(x)y + x(d + g)(y) \quad \forall \ x, y \in S.
\]

Therefore \(d + g\) is a derivation on \(S\).

**Theorem 3.11.** [Linearization Theorem] Let \(S\) be a semiring and \(X\) a \(S\)-module. Let \((d, g)\) be a Jordan \(S\)-derivation pair on \(S\). Then for all \(x, y \in S\) the following statements hold:
\[
d(xy + yx) = d(x)y + d(y)x + xg(y) + yg(x),
\]
\[
g(xy + yx) = g(x)y + g(y)x + xd(y) + yd(x).
\]

**Proof.** Since \((d, g)\) is a Jordan \(S\)-derivation pair on
\[
S, d(x^2) = d(x)x + xg(x) \quad \forall \ x \in S.
\] (1)
Replacing \(x\) by \(x + y\) in (1), we get
\[
d((x + y)^2) = d(x + y)(x + y) + (x + y)g(x + y) \quad \forall \ x, y \in S
\] (2)
\[
\quad = [d(x) + d(y)][x + y] + (x + y)[g(x) + g(y)]
\]
\[
d((x + y)^2) = d(x)x + d(y)x + d(x)y + d(y)y + xg(x) + xg(y) + yg(x)
\]
\[
\quad + yg(y) \quad \forall \ x, y \in S
\] (3)

However,
\[
d(x^2 + xy + yx + y^2) = d(x^2) + d(xy + yx) + d(y^2) \quad \forall \ x, y \in S
\]
\[
d((x + y)^2) = d(x)x + xg(x) + d(xy + yx) + d(y)y
\]
\[
\quad + yg(y) \quad \forall \ x, y \in S
\] (4)
Comparing (3) and (4), we have
\[ d(xy + yx) = d(y)x + xg(y) + d(x)y + xg(y) \quad \forall \ x, y \in S \]
Similarly, we can obtain
\[ g(xy + yx) = g(x)y + g(y)x + xd(y) + yd(x) \quad \forall \ x, y \in S. \]

**Theorem 3.12.** Let \( S \) be an additively cancellative semiring and \( X \) a \( S \)--module. Let \((d, g)\) be a Jordan \( S \)--derivation pair on \( S \) such that \( d(1) = g(1) \). Then \( d(x) = g(x) \quad \forall \ x \in S \).

*Proof.* Define a mapping \( f : S \to X \) by \( f(x) = d(x) - g(x) \quad \forall \ x \in S \).
Since \((d, g)\) is a Jordan \( S \)--derivation pair on \( S \),
\[ d(x^2) = d(x)x + xg(x) \quad \forall \ x \in S \quad (1) \]
\[ g(x^2) = g(x)x + xd(x) \quad \forall \ x \in S \quad (2) \]
From (1) and (2), we get
\[ (d - g)(x^2) = (d - g)(x)x + x(g - d)(x) \quad \forall \ x \in S \]
\[ f(x^2) = f(x)x + x(-f(x)) \quad \forall \ x \in S \]
\[ f(x^2) = f(x)x - xf(x) \quad \forall \ x \in S \quad (3) \]
Linearization of (3), gives
\[ f(xy + yx) = f(x)y - xf(y) + f(y)x - yf(x) \quad \forall \ x, y \in S \quad (4) \]
Since \( f(1) = 0 \), and put \( y = 1 \) in (4), we have
\[ 2f(x) = f(x) - f(x) \Rightarrow f(x) = 0 \Rightarrow d(x) = g(x) \quad \forall \ x \in S. \]

**Theorem 3.13.** Let \( S \) be a semiring and \( X \) a \( S \)--module. Let \((d, g)\) be a Jordan \( S \)--derivation pair on \( S \). Then \( d \) and \( g \) are Jordan derivations.

*Proof.* Since \((d, g)\) be a Jordan \( S \)--derivation pair on \( S \), we have
\[ d(x^2) = d(x)x + xg(x) \quad \forall \ x \in S \]
and
\[ g(x^2) = g(x)x + xd(x) \quad \forall \ x \in S. \]
Using above theorem, we get
\[ d(x^2) = d(x)x + xd(x) \quad \forall \ x \in S \]
and
\[
g(x^2) = g(x)x + xg(x) \quad \forall \ x \in S
\]
Hence \(d\) and \(g\) are Jordan derivations on \(S\).

With similar arguments, we can establish the following

**Theorem 3.14.** Let \(S\) be a semiring and \(X\) a \(S\)-module. Let \((d, g)\) be a \(S\)-derivation pair on \(S\). Then \(d\) and \(g\) are derivations.

**References**


