On (2, 2)-Domination in Hexagonal Networks

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Abstract

A subset $S$ of the vertex set of a graph $G$ is a $(2, 2)$-dominating set if for every vertex $v \in V(G) \setminus S$ there exist at least 2 vertices in $S$ which are at a distance at most 2 from $v$. The parameter $\gamma_{2,2}(G)$ denotes the minimum cardinality of a minimal $(2, 2)$-dominating set of $G$ and it gives the $(2, 2)$-dominating number of $G$. In this paper, we obtain an upper bound for the $(2, 2)$-dominating number of a popular mesh-derived parallel architecture called the Hexagonal networks.

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1 Introduction

The study of domination and its variants in graphs is a rapidly developing area of research in graph theory due to its various applications to adhoc networks, web graphs, distributed computing and social networks [12, 13]. A dominating set $S$ of a graph $G$ is a vertex subset with the property that every vertex of $G$ that is not in $S$ is adjacent to a vertex in $S$. The minimum size of the dominating set is called the domination number and is denoted by $\gamma(G)$. Let $G = (V(G), E(G))$ be a simple graph. For $x, y \in V(G)$, the distance $d_G(x, y)$ between $x$ and $y$ is the length of the shortest $xy$-paths in $G$. The diameter of $G$ is $d(G) = \max \{d_G(x, y) : x, y \in V(G)\}$. For an integer $k \geq 1$ and $x \in V(G)$, the open $k$-neighborhood of $x$ is $N_k(x) = \{y \in V(G) : 0 < d_G(x, y) \leq k\}$, and the closed $k$-neighborhood of $x$ is $N_k[x] = N_k(x) \cup \{x\}$. For a vertex $x \in V(G)$ we define the degree of $x$ as $d(x) = |N(x)|$. The minimum degree of $G$ is denoted by $\delta(G) = \min\{d(x) : x \in V(G)\}$. For any positive integer $k$ and any graph $G$ the $k$-th power $G^k$ of $G$ is the graph with vertex set $V(G)$ where two different vertices are adjacent if and only if the distance between them is at
most $k$ in $G$. Furthermore, the minimum $k$-degree $\delta_k(G)$ of $G$ is defined by $\delta_k(G) = \delta(G^k)$.

For two positive integers $k$ and $r$, a subset $S$ of the vertex of a graph $G$ is $(k,r)$-dominating set of $G$ if every vertex $x \in V(G) \setminus S$ is within distance $k$ to at least $r$ vertices in $S$. The parameter $\gamma_{k,r}(G)$ denotes the minimum cardinality of a $(k,r)$-dominating set of $G$ and is called the $(k,r)$-dominating number. The open $r$-neighbourhood $N_r(v)$ of a vertex $v$ in a graph $G$ is defined by $N_r(v) = \{u \in V(G) : 0 < d(u,v) \leq r\}$ and its closed $r$-neighbourhood is $N_r[v] = N_r(v) \cup \{v\}$. The $r$-degree of $v$ in $G$, $\operatorname{degr}(v)$, is given by $|N_r(v)|$, while $\Delta_r(G)$ and $\delta_r(G)$ denote the maximum and minimum $r$-degree among all the vertices of $G$ respectively [6].

This dominating concept is a generalization of the two concepts distance domination and $r$-domination. So the study of $(k,r)$-domination of a graph is more interesting and has received the attention of many researchers. In this paper, we consider the $(2,2)$-domination problem for hexagonal networks.

2 Hexagonal Networks

Parallel processing is considered as one of the hot topics in the computing era. The fact that interconnection networks form the heart of parallel processing has motivated many researchers to create, evaluate, explore and improve many interconnection network topologies [1, 3, 4, 7]. It is difficult to design a network that is ideal from all aspects. One has to design a suitable network depending on its properties and requirements. Thus many graphs are proposed as possible interconnection network topologies. It is well known that triangular, square and hexagonal are the three regular plane tessellations, composed of the same kind of regular polygons. They are the foundation for the design of direct interconnection networks which are highly competitive in overall performance. Grid connected computers and tori are based on regular square tessellations, and are popular and well-known models for parallel processing.

Hexagonal networks $HX(n)$ are multiprocessor interconnection network based on regular triangular tessellations and this is widely studied in [10]. An $n$-dimensional hexagonal mesh $HX(n)$ has $3n^2 - 3n + 1$ vertices and $9n^2 - 15n + 6$ edges where $n$ is the number of vertices on one side of the hexagon [9]. The diameter of $HX(n)$ is $2(n-1)$. There are six vertices of degree three which we call as corner vertices $\{\alpha, \beta, \gamma, \delta, \eta, \sigma\}$. There is a unique vertex which is at distance $n-1$ from each of the corner vertices called the center of $HX(n)$ and is represented by $O$. The vertex set $V$ is partitioned into sets including concentric cycles around $O$. Call vertex $O$ as level 1, the first cycle around $O$ as level 2 denoted by $C^O_2$ and so on. The last cycle farthest from $O$ is level $n$.
denoted by $C_n^O$. The level $i$ cycle has $6(i-1)$ vertices $i \geq 2$ [2]. See Figure 1.

![Figure 1: A hexagonal mesh of dimension 5, $HX(5)$.](image1)

In [8], Stojmenovic suggested a coordinate system for a honeycomb network. This was revised by Nocetti et al. [5] to assign coordinates to the vertices in the hexagonal network. In this scheme, three axes, $X$, $Y$ and $Z$ parallel to three edge directions and at mutual angle of 120 degrees between any two of them are introduced, as indicated in Figure 2. We call lines parallel to the coordinate axes as $X$-lines, $Y$-lines and $Z$-lines. Here $X = h$ and $X = -k$ are two $X$-lines on either side of the $X$-axis. Any vertex of $HX(n)$ is assigned coordinates $(x, y, z)$ in the above scheme.

![Figure 2: Coordinates of vertices in $HX(5)$.](image2)

Hexagonal networks has been studied in a variety of contexts and has been applied in cellular networks [5], computer graphics [9], image processing and chemistry to model benzenoid hydrocarbons [11].

3 The $(2,2)$-domination number of hexagonal networks

The following theorems are some known results and bounds for $(k,r)$-domination in graphs.
Theorem 1. [6] Let $G$ be a graph. Then $k \leq \gamma_{k,r}(G) \leq n$ and these bounds are sharp.

Theorem 2. [6] If $r = \text{diam}(G)$, then $\gamma_{k,r}(G) = k$.

Theorem 3. [6] If $G$ is a spider with $n$ vertices, then $\gamma_{2,2}(G) = (n+1)/2$.

Corollary 4. [6] If $G$ is a connected graph on $n \geq 3$ vertices, then $\gamma_{2,2}(G) = (n+1)/2$ with equality if and only if $G$ is a spider.

The following theorems gives an upper bound for $(2,2)$-domination number of hexagonal networks.

Proposition 5. For a 2-dimensional hexagonal mesh $HX(2)$, $\gamma_{2,2}(HX(2)) = 2$.

Proof. Since the diameter of $HX(2)$ is 2, by theorem 2, $\gamma_{2,2}(HX(2)) = 2$.

Remark: In this case, the minimal $(2,2)$-dominating set can consist of any two of the vertices of $HX(2)$.

Theorem 6. For any $n$-dimensional hexagonal mesh $HX(2)$ ($n > 2$), if $n$ is even then there exists a $(2,2)$-dominating set $D$ such that $|D| = (n^2 - 2)/2$.

Proof. For any given $n$-dimensional hexagonal mesh $HX(2)$ ($n > 2$), we choose a subset $D$ of the vertex set $V$ as follows; the set $D$ contains (i) the center vertex $O$ of the given $HX(2)$, (ii) the vertices of the cycles $C_{O}^{i}$, ($4 \leq i \leq n$ and $i$ is even), starting with the corner vertex with the label $(i-1,0,-(i-1))$ such that between any two of these vertices there is a path of length 3 on the cycle. Figure 3 shows the vertices of $D$ for $HX(8)$. We claim that $D$ is a $(2,2)$-dominating set such that $|D| = (n^2 - 2)/2$. The proof is by induction on the dimension $n$ ($n > 2$ and $n$ is even).

Figure 3: Construction of set $D$ in $HX(8)$.

If $n = 4$, the set $D$ will consist of the centre vertex $O$ (i.e. vertex with label $(0,0,0)$) of $HX(4)$ and the six vertices on the boundary of $HX(4)$, starting
with the corner vertex with label \((3, 0, -3)\) such that between any two of these vertices there is a path of length 3 on the boundary. See Figure 4.(a). Since the length of the path between any two of these six vertices is three the remaining vertices on the boundary will be at a distance at most two from exactly two of these vertices. So it remains to show that the vertices of the cycles \(C_2^O\) and \(C_3^O\) (the vertices on the dotted lines in Figure 4.(b)) are at a distance at most two from at least two of the vertices in \(D\). Clearly these vertices are at distance at most two from the center. Also, the union of 2-neighbourhoods of the vertices of \(D\) except the centre vertex contains all the vertices the vertices of the cycles \(C_2^O\) and \(C_3^O\). Hence \(D\) is a \((2, 2)\)-dominating set with \(|D| = 7 = (4^2 - 2)/2 = (n^2 - 2)/2\).

**Figure 4:** (a) and (b) Illustration for the case \(n=4\).

Assume the result is true for \(HX(n)\). Consider a hexagonal mesh of dimension \(n + 2\), \(HX(n + 2)\). Let \(D'\) be the subset of its vertex set as defined in the beginning. \(HX(n + 2)\) consist of cycles \(C_i^O\), \(2 \leq i \leq n + 2\) having the same center \(O\). By induction, there exist a \((2, 2)\)-dominating set of \(HX(n)\), say \(D\) such that \(|D| = (n^2 - 2)/2\). Evidently the set \(D'\) is the union of the set \(D\) and the vertices on the boundary of \(HX(n + 2)\) starting with the corner vertex with the label \((n + 1, 0, -(n + 1))\) such that between any two of these vertices there is a path of length 3 on the boundary. Now we will show that \(D'\) is a \((2, 2)\)-dominating set of \(HX(n + 2)\). The vertices of \(D'\) that lie on the boundary of \(HX(n + 2)\) are chosen such that between any two of these vertices there is a path of length 3 on the boundary. Hence the remaining vertices on the boundary of \(HX(n + 2)\) are at a distance at most two from exactly two of these vertices. Also the set \(D\) is a \((2, 2)\)-dominating set of \(HX(n)\). Hence it will dominate all the vertices of cycles \(C_i^O\), \(2 \leq i \leq n\). Now it remains to show that all the vertices on the boundary of cycles \(C_{n+1}^O\) will be at a distance at most two from at least two of the vertices in \(D'\). We observe that we can find two types of trapezoidal structure as show in Figure 6 between these vertices and the vertices of \(D'\) on \(C_{n+1}^O\), such that their union will contain all the vertices of \(C_{n+1}^O\). Clearly from the Figure 5 (a) that in both these trapezoidal structures the vertices of \(C_{n+1}^O\) will be at a distance at most two from at least one of the vertices in \(D'\) that lie on the boundary of \(HX(n)\).
Similarly we can also find two other types of trapezoidal structure as Figure 5.(b) between the vertices of \( C_{n+1}^O \) and the vertices of \( D' \) on the boundary of \( HX(n+2) \) such that the vertices of \( C_{n+1}^O \) will be at a distance at most two from at least one of the vertices in \( D' \) that lie on the boundary of \( HX(n+2) \). Hence all the vertices of \( C_{n+1}^O \) will be at a distance at most two from at least two of the vertices in \( D' \). Hence \( D' \) is a \((2,2)\)-dominating set of \( HX(n+2) \).

By the choice of vertices of \( D' \), the number of vertices in \( D' \) that lie on the boundary of \( HX(n+2) \) is \((6(n+2) - 6)/3 = 2(n+1)\). Hence \(|D'| = |D| + 2(n+1) = (n+2)^2 - 2)/2 \).

**Theorem 7.** For any \( n \)-dimensional hexagonal mesh \( HX(2) \) \((n > 2)\), if \( n \) is odd then there exists a \((2,2)\)-dominating set \( D \) such that \(|D| = (n^2 + 1)/2\).

**Proof.** For any given \( n \)-dimensional hexagonal mesh \( HX(2) \) \((n > 2)\), we will choose a subset \( D \) of the vertex set such that it consists of the following vertices, (i) the centre vertex \( O \) of the given \( HX(2) \), (ii) the vertices of the cycles \( C_i^O \), \((3 \leq i \leq 8 \text{ and } i \text{ is even})\) starting with the corner vertex with the label \((i-1,0,-(i-1))\) such that between any two of these vertices there is a path of length 3 on the boundary. Figure 6, shows the vertices of \( D \) for \( HX(5) \). We must prove that \( D \) is a \((2,2)\)-dominating set such that \(|D| = (n^2 + 1)/2\).

The proof is similar to that of theorem 6.

**Figure 6:** \((2,2)\)-domination of \( HX(7) \).

In the view of theorems 6 and 7 we imply the following,

**Theorem 8.** For any \( n \)-dimensional hexagonal mesh \( HX(n) \) \((n > 2)\),

\[
\gamma_{2,2}(HX(n)) \leq \begin{cases} 
(n^2 - 2)/2, & \text{when } n \text{ is even} \\
(n^2 + 1)/2, & \text{when } n \text{ is odd}.
\end{cases}
\]

**Remark:** It is interesting to note that theorem 8 provides an improved upper bound for \((2,2)\)-domination number of hexagonal networks. Figure 7
exhibits that the newly obtained upper bound for $(2,2)$-domination number of hexagonal networks is significantly smaller when compared to the general upper bound for any connected graph obtained in corollary 4.

![Comparison of upper bounds](image)

Figure 7: Comparison of the upper bounds for $(2,2)$-domination number of $HX(n)$

4 Conclusion

In this paper we obtain an upper bond for the $(2,2)$-domination number of hexagonal mesh. We observe that in the case of hexagonal mesh we get a significantly reduced upper bound when compared to the general upper bound for any connected graphs. Finding a tight lower bound for $(2,2)$-domination number of hexagonal mesh is quite challenging. It would be interesting to explore tight bounds for other interconnection networks too.

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References


