

PROPERTIES ON CERTAIN SUBCLASS OF GENERALIZED SAKAGUCHI TYPE FUNCTIONS

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Abstract: The object of the present paper is to determine coefficient estimates, neighborhoods and partial sums for functions $f(z)$ belonging to $k\widetilde{\mathcal{US}}_s^\eta(\rho, \mu, \gamma, t)$.

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1. Introduction

Let $\mathcal{A}$ denote the family of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

that are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Now, by making use of the differential operator $D_{\rho, \mu}^\eta$, (see [1] and [2]) we define a new subclass of functions belonging to the class $\mathcal{A}$.

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Definition 1.1. A function $f(z) \in \mathcal{A}$ is said to be in the class $k\text{-}\mathcal{US}_S^\eta(\rho, \mu, \gamma, t)$ if for all $z \in \mathcal{U}$,

$$\begin{aligned} & \operatorname{Re} \left\{ \frac{(1-t)[\rho z^2(D_{\rho,\mu}^\eta f(z))'' + z(D_{\rho,\mu}^\eta f(z))']}{\rho z[(D_{\rho,\mu}^\eta f(z))' - t(D_{\rho,\mu}^\eta f(tz))'] + (1-\rho)[(D_{\rho,\mu}^\eta f(z)) - (D_{\rho,\mu}^\eta f(tz))]} \right\} \\ & \geq k \left| \frac{(1-t)[\rho z^2(D_{\rho,\mu}^\eta f(z))'' + z(D_{\rho,\mu}^\eta f(z))']}{\rho z[(D_{\rho,\mu}^\eta f(z))' - t(D_{\rho,\mu}^\eta f(tz))'] + (1-\rho)[(D_{\rho,\mu}^\eta f(z)) - (D_{\rho,\mu}^\eta f(tz))]} - 1 \right| \\ & \quad + \gamma, \end{aligned}$$

for $\rho \geq \mu \geq 0$, $\eta, k \geq 0$, $|t| \leq 1$, $t \neq 1$, $0 \leq \gamma < 1$.

Furthermore, we say that a function $f(z) \in k\text{-}\mathcal{US}_S^\eta(\rho, \mu, \gamma, t)$ is in the sub-class $k\text{-}\widetilde{\mathcal{US}}_S^\eta(\rho, \mu, \gamma, t)$ if $f(z)$ is of the following form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0, n \in \mathbb{N}). \quad (1.2)$$

The aim of the present paper is to study the coefficient bounds, partial sums and certain neighborhood results of the class $k\text{-}\widetilde{\mathcal{US}}_S^\eta(\rho, \mu, \gamma, t)$.

Let $A(z) = (1-t)[\rho z^2(D_{\rho,\mu}^\eta f(z))'' + z(D_{\rho,\mu}^\eta f(z))']$
and $B(z) = \rho z[(D_{\rho,\mu}^\eta f(z))' - t(D_{\rho,\mu}^\eta f(tz))'] + (1-\rho)[(D_{\rho,\mu}^\eta f(z)) - (D_{\rho,\mu}^\eta f(tz))]$.

2. Coefficient Bounds of the Function Class $k\text{-}\widetilde{\mathcal{US}}_S^\eta(\rho, \mu, \gamma, t)$

Firstly, we shall need the following lemmas.

Lemma 2.1. Let $w = u + iv$. The $\operatorname{Re} w \geq \alpha$ if and only if $|w - (1 + \alpha)| \leq |w + (1 - \alpha)|$.

Lemma 2.2. Let $w = u + iv$ and α, γ are real numbers. Then $\operatorname{Re} w > \alpha|w - 1| + \gamma$ if and only if $\operatorname{Re}\{w(1 + \alpha e^{i\theta}) - \alpha e^{i\theta}\} > \gamma$.

Theorem 2.1. The function $f(z)$ defined by (1.2) is in the class $k\text{-}\widetilde{\mathcal{US}}_S^\eta(\rho, \mu, \gamma, t)$ if and only if

$$\begin{aligned} & \sum_{n=2}^{\infty} \phi^\eta(\rho, \mu, n) |\rho n^2(k+1) - n\rho(2k + \gamma + 1) + n(1+k) - u_n[(k + \gamma) \\ & \quad - \rho(k + \gamma) + n\rho t(k + \gamma)]| a_n \leq 1 - \gamma(\rho t + 1) - \rho t k \end{aligned} \quad (2.1)$$

where $\rho \geq \mu \geq 0$, $\eta, k \geq 0$, $|t| \leq 1$, $t \neq 1$, $0 \leq \gamma < 1$, $u_n = 1 + t + \dots + t^{n-1}$.
The result is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{1 - \gamma(\rho t + 1) - \rho t k}{\left\{ \phi^\eta(\rho, \mu, n) [\rho n^2(k + 1) - n\rho(2k + \gamma + 1) + n(1 + k) - u_n[(k + \gamma) - \rho(k + \gamma) + n\rho t(k + \gamma)]] \right\}} z^n$$

Proof. By Definition 1.1, we get

$$Re \left\{ \frac{A(z)}{B(z)} \right\} \geq k \left| \frac{A(z)}{B(z)} - 1 \right| + \gamma$$

Then by Lemma 2.2, we have

$$Re \left\{ \frac{A(z)}{B(z)} (1 + ke^{i\theta}) - ke^{i\theta} \right\} \geq \gamma, -\pi < \theta \leq \pi$$

or equivalently

$$Re \left\{ \frac{A(z)(1 + ke^{i\theta}) - ke^{i\theta} B(z)}{B(z)} \right\} \geq \gamma. \quad (2.2)$$

Let $F(z) = A(z)(1 + ke^{i\theta}) - ke^{i\theta} B(z)$ and $E(z) = B(z)$.
By Lemma 2.1, (2.2) is equivalent to

$$|F(z) + (1 - \gamma)E(z)| \geq |F(z) - (1 + \gamma)E(z)|, \text{ for } 0 \leq \gamma < 1.$$

But

$$\begin{aligned} & |F(z) + (1 - \gamma)E(z)| \\ &= \left| (1 - t) \left\{ (2 - \gamma(\rho t + 1) + \rho t(1 - ke^{i\theta}))z \right. \right. \\ &\quad - \sum_{n=2}^{\infty} \phi^\eta(\rho, \mu, n)(n[\rho(n - \gamma) + 1] + (1 - \gamma)u_n(1 - \rho + n\rho t))a_n z^n \\ &\quad \left. \left. - ke^{i\theta} \sum_{n=2}^{\infty} \phi^\eta(\rho, \mu, n)(n[\rho(n - 2) + 1] - u_n(1 - \rho + n\rho t))a_n z^n \right\} \right| \\ &\geq |1 - t| \left\{ (2 - \gamma(\rho t + 1) + \rho t(1 - k))|z| \right. \\ &\quad \left. - \sum_{n=2}^{\infty} \phi^\eta(\rho, \mu, n)(n[\rho(n - \gamma) + 1] + (1 - \gamma)u_n(1 - \rho + n\rho t))|a_n|z|^n \right\} \end{aligned}$$

$$-k \sum_{n=2}^{\infty} \phi^n(\rho, \mu, n) |(n[\rho(n-2) + 1] - u_n(1 - \rho + n\rho t))| a_n |z|^n \Big\}.$$

Also

$$\begin{aligned} & |F(z) - (1 + \gamma)E(z)| \\ &= \left| (1-t) \left\{ [-\gamma(\rho t + 1) - \rho t(1 + ke^{i\theta})]z \right. \right. \\ &\quad - \sum_{n=2}^{\infty} \phi^n(\rho, \mu, n) (n[\rho(n-2 - \gamma) + 1] - (1 + \gamma)u_n(1 - \rho + n\rho t)) a_n z^n \\ &\quad \left. \left. - ke^{i\theta} \sum_{n=2}^{\infty} \phi^n(\rho, \mu, n) (n[\rho(n-2) + 1] - u_n(1 - \rho + n\rho t)) a_n z^n \right\} \right| \\ &\leq |1-t| \left\{ (\gamma(\rho t + 1) - \rho t(1 + k))|z| \right. \\ &\quad + \sum_{n=2}^{\infty} \phi^n(\rho, \mu, n) |(n[\rho(n-2 - \gamma) + 1] - (1 + \gamma)u_n(1 - \rho + n\rho t))| a_n |z|^n \\ &\quad \left. + k \sum_{n=2}^{\infty} \phi^n(\rho, \mu, n) |(n[\rho(n-2) + 1] - u_n(1 - \rho + n\rho t))| a_n |z|^n \right\} \end{aligned}$$

and so

$$\begin{aligned} & |F(z) + (1 - \gamma)E(z)| - |F(z) - (1 + \gamma)E(z)| \\ &\geq |1-t| \left\{ 2(1 - \gamma(\rho t + 1) - \rho tk)|z| \right. \\ &\quad \left. - \sum_{n=2}^{\infty} \left\{ \begin{aligned} & \phi^n(\rho, \mu, n) [|n[\rho(n - \gamma) + 1] + (1 - \gamma)u_n(1 - \rho + n\rho t)| \\ & + |(n[\rho(n-2) + 1] - u_n(1 + \gamma)(1 - \rho + n\rho t))| \\ & + 2k|n[\rho(n-2) + 1] - u_n(1 - \rho + n\rho t)| a_n |z|^n \end{aligned} \right\} \right\} \\ &\geq 2(1 - \gamma(\rho t + 1) - \rho tk)|z| \\ &\quad - \sum_{n=2}^{\infty} \left\{ \begin{aligned} & 2\phi^n(\rho, \mu, n) |\rho n^2(k+1) - n\rho(2k + \gamma + 1) + n(k+1)| \\ & - u_n[(k + \gamma) - \rho(k + \gamma) + n\rho t(k + \gamma)] |a_n| |z|^n \end{aligned} \right\} \geq 0 \end{aligned}$$

or

$$\begin{aligned} & \sum_{n=2}^{\infty} \left\{ \begin{aligned} & \phi^n(\rho, \mu, n) |\rho n^2(k+1) - n\rho(2k + \gamma + 1) + n(k+1)| \\ & - u_n[(k + \gamma) - \rho(k + \gamma) + n\rho t(k + \gamma)] |a_n| \end{aligned} \right\} \\ & \leq 1 - \gamma(\rho t + 1) - \rho tk. \end{aligned}$$

Conversely, suppose that (2.1) holds. Then we must show

$$Re \left\{ \frac{A(z)(1 + ke^{i\theta}) - ke^{i\theta}(B(z))}{B(z)} \right\} \geq \gamma.$$

Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, the above inequality reduces to

$$Re \left\{ \frac{(1 - \gamma(\rho t + 1) - \rho t k) - \sum_{n=2}^{\infty} \left\{ \phi^{\eta}(\rho, \mu, n)(\rho n^2(1 + ke^{i\theta}) - n\rho(2ke^{i\theta} + \gamma + 1) + n(1 + ke^{i\theta}) - u_n[(ke^{i\theta} + \gamma) - \rho(ke^{i\theta} + \gamma) + n\rho t(ke^{i\theta} + \gamma)])a_n z^{n-1} \right\}}{(\rho t + 1) - \sum_{n=2}^{\infty} \phi^{\eta}(\rho, \mu, n)(n\rho + u_n(1 - \rho + n\rho t))a_n z^{n-1}} \right\} \geq 0.$$

Since $Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1$, the above inequality reduces to

$$Re \left\{ \frac{(1 - \gamma(\rho t + 1) - \rho t k) - \sum_{n=2}^{\infty} \left\{ \phi^{\eta}(\rho, \mu, n)(\rho n^2(k + 1) - n\rho(2k + \gamma + 1) + n(k + 1) - u_n[(k + \gamma) - \rho(k + \gamma) + n\rho t(k + \gamma)])a_n r^{n-1} \right\}}{(\rho t + 1) - \sum_{n=2}^{\infty} \phi^{\eta}(\rho, \mu, n)(n\rho + u_n(1 - \rho + n\rho t))a_n r^{n-1}} \right\} \geq 0.$$

Letting $r \rightarrow 1^-$, we have desired conclusion. $\square$

Corollary 2.1. *If $f(z) \in k\widetilde{\mathcal{US}}_S^{\eta}(\rho, \mu, \gamma, t)$, then*

$$a_n \leq \frac{1 - \gamma(\rho t + 1) - \rho t k}{\left\{ \phi^{\eta}(\rho, \mu, n)|\rho n^2(k + 1) - n\rho(2k + \gamma + 1) + n(k + 1) - u_n[(k + \gamma) - \rho(k + \gamma) + n\rho t(k + \gamma)] \right\}},$$

where $\rho \geq \mu \geq 0$, $\eta, k \geq 0$, $|t| \leq 1$, $t \neq 1$, $0 \leq \gamma < 1$, $u_n = 1 + t + \dots + t^{n-1}$.

3. Neighborhood of the Function Class $k\widetilde{\mathcal{US}}_S^\eta(\rho, \mu, \gamma, t)$

Following the earlier investigations (based upon the familiar concept of neighborhoods of analytic functions) by Goodman [3], Ruscheweyh [4].

Definition 3.1. Let $\rho \geq \mu \geq 0$, $\eta, k \geq 0$, $|t| \leq 1$, $t \neq 1$, $0 \leq \gamma < 1$, $\alpha \geq 0$, $u_n = 1 + t + \cdots + t^{n-1}$. We define the α -neighborhood of a function $f \in \mathcal{A}$ and denote by $N_\alpha(f)$ consisting of all functions $g(z) = z - \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}$ ($b_n \geq 0$, $n \in \mathbb{N}$) satisfying

$$\sum_{n=2}^{\infty} \frac{\{\phi^\eta(\rho, \mu, n)[pn^2(k+1) - n\rho(2k+\gamma+1) + n(k+1) - u_n[(k+\gamma) - \rho(k+\gamma) + npt(k+\gamma)]]\}}{1 - \gamma(\rho t + 1) - \rho t k} |a_n - b_n| \leq \alpha.$$

Theorem 3.1. Let $f \in k\widetilde{\mathcal{US}}_S^\eta(\rho, \mu, \gamma, t)$ and for all real θ we have $\gamma(e^{i\theta} - 1) - 2e^{i\theta} \neq 0$. For any complex number ϵ with $|\epsilon| < \alpha$ ($\alpha \geq 0$), if f satisfies the following condition $\frac{f(z)+\epsilon z}{1+\epsilon} \in k\widetilde{\mathcal{US}}_S^\eta(\rho, \mu, \gamma, t)$, then $N_\alpha(f) \subset k\widetilde{\mathcal{US}}_S^\eta(\rho, \mu, \gamma, t)$.

Proof. It is obvious that $f \in k\widetilde{\mathcal{US}}_S^\eta(\rho, \mu, \gamma, t)$ if and only if

$$\left| \frac{A(z)(1 + ke^{i\theta}) - (ke^{i\theta} + 1 + \gamma)(B(z))}{A(z)(1 + ke^{i\theta} - (1 - ke^{i\theta} - \gamma)(B(z)))} \right| < 1,$$

($-\pi < \theta < \pi$) for any complex number s with $|s| = 1$, we have

$$\frac{A(z)(1 + ke^{i\theta}) - (ke^{i\theta} + 1 + \gamma)(B(z))}{A(z)(1 + ke^{i\theta} - (1 - ke^{i\theta} - \gamma)(B(z)))} \neq s$$

In other words, we must have

$$(1-s)(1-t)(A(z))(1 + ke^{i\theta}) - (ke^{i\theta} + 1 + \gamma + s(1 - ke^{i\theta} - \gamma))(B(z)) \neq 0$$

which is equivalent to

$$z - \sum_{n=2}^{\infty} \frac{\{\phi^\eta(\rho, \mu, n)[(\rho n^2 - \rho n + n) - (n\rho + u_n(1 - \rho + n\rho t))](1 + ke^{i\theta} - ske^{i\theta}) - s[(\rho n^2 - \rho n + n) + (n\rho + u_n(1 - \rho + n\rho t))]\}}{(1-s)(1 + ke^{i\theta} - ke^{i\theta}(\rho t + 1)) + (\rho t + 1)(-(1+s) + \gamma(s-1))} \neq 0.$$

However, $f \in k\widetilde{\mathcal{US}}_S^\eta(\rho, \mu, \gamma, t)$ if and only if $\frac{(f*h)(z)}{z} \neq 0$, $z \in \mathcal{U} - \{0\}$, where $h(z) = z - \sum_{n=2}^{\infty} c_n z^n$, and

$$C_n = \frac{\left\{ \phi^\eta(\rho, \mu, n) [(\rho n^2 - \rho n + n) - (n\rho + u_n(1 - \rho + n\rho t))](1 + ke^{i\theta} - ske^{i\theta}) - s[(\rho n^2 - \rho n + n) + (n\rho + u_n(1 - \rho + n\rho t))] - \gamma(1 - s)[n\rho + u_n(1 - \rho + n\rho t)] \right\}}{(1 - s)(1 + ke^{i\theta} - ke^{i\theta}(\rho t + 1)) + (\rho t + 1)(-(1 + s) + \gamma(s - 1))}$$

We note that

$$|C_n| \leq \frac{\left\{ \phi^\eta(\rho, \mu, n) |\rho n^2(1 + k) - n\rho(2k + \gamma + 1) + n(k + 1) - u_n[(k + \gamma) - \rho(k + \gamma) + n\rho t(k + \gamma)] \right\}}{1 - \gamma(\rho t + 1) - \rho t k}$$

Since $\frac{f(z)+\epsilon z}{1+\epsilon} \in k\widetilde{\mathcal{US}}_S^\eta(\rho, \mu, \gamma, t)$, therefore $z^{-1} \left(\frac{f(z)+\epsilon z}{1+\epsilon} * h(z) \right) \neq 0$, which is equivalent to

$$\frac{(f * h)(z)}{(1 + \epsilon)z} + \frac{\epsilon}{1 + \epsilon} \neq 0 \quad (3.1)$$

Now suppose that $\left| \frac{(f*h)(z)}{z} \right| < \alpha$. Then by (3.1), we must have

$$\left| \frac{(f * h)(z)}{(1 + \epsilon)z} + \frac{\epsilon}{1 + \epsilon} \right| \geq \frac{|\epsilon|}{|1 + \epsilon|} - \frac{1}{|1 + \epsilon|} \left| \frac{(f * h)(z)}{z} \right| > \frac{|\epsilon| - \alpha}{|1 + \epsilon|} \geq 0,$$

this is a contradiction by $|\epsilon| < \alpha$ and however, we have $\left| \frac{(f*h)(z)}{z} \right| \geq \alpha$. If

$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \in N_\alpha(f)$, then

$$\begin{aligned} \alpha - \left| \frac{(g * h)(z)}{z} \right| &\leq \left| \frac{((f - g) * h)(z)}{z} \right| \leq \sum_{n=2}^{\infty} |a_n - b_n| |c_n| |z^n| \\ &< \sum_{n=2}^{\infty} \frac{\left\{ \phi^\eta(\rho, \mu, n) |\rho n^2(1 + k) - n\rho(2k + \gamma + 1) + n(k + 1) - u_n[(k + \gamma) - \rho(k + \gamma) + n\rho t(k + \gamma)] \right\}}{1 - \gamma(\rho t + 1) - \rho t k} \\ &\times |a_n - b_n| \leq \alpha. \end{aligned} \quad \square$$

4. Partial Sums of the Function Class $k\widetilde{\mathcal{US}}_S^\eta(\rho, \mu, \gamma, t)$

In this section applying method used by Silverman [5], we investigate the ratio of a function of the form (1.2) to its sequence of partial sums $f_m(z) = z + \sum_{n=2}^m a_n z^n$.

Theorem 4.1. *If f of the form (1.1) satisfies the condition (2.1), then*

$$\operatorname{Re} \left\{ \frac{f(z)}{f_m(z)} \right\} \geq 1 - \frac{1}{\delta_{m+1}} \quad (4.1)$$

and

$$\delta_n \geq \begin{cases} 1, & n = 2, 3, \dots, m \\ \delta_{m+1}, & n = m+1, m+2, \dots \end{cases} \quad (4.2)$$

where

$$\delta_n = \frac{\{\phi^n(\rho, \mu, n) | \rho n^2(k+1) - n\rho(2k+\gamma+1) + n(k+1) - u_n[(k+\gamma) - \rho(k+\gamma) + n\rho t(k+\gamma)] | \}}{1 - \gamma(\rho t + 1) - \rho t k}. \quad (4.3)$$

The result in (4.1) is sharp for every m , with the extremal function

$$f(z) = z + \frac{z^{m+1}}{\delta_{m+1}}. \quad (4.4)$$

We next determine bounds for $f_m(z)/f(z)$.

Theorem 4.2. *If f of the form (1.1) satisfies the condition (2.1), then*

$$\operatorname{Re} \left\{ \frac{f_m(z)}{f(z)} \right\} \geq \frac{\delta_{m+1}}{1 + \delta_{m+1}}. \quad (4.5)$$

The result is sharp with the function given by (4.4).

We next turn to ratios involving derivatives.

Theorem 4.3. *If f of the form (1.1) satisfies the condition (2.1), then*

$$\operatorname{Re} \left\{ \frac{f'(z)}{f'_m(z)} \right\} \geq 1 - \frac{m+1}{\delta_{m+1}}, \quad (4.6)$$

$$Re \left\{ \frac{f'_m(z)}{f'(z)} \right\} \geq \frac{\delta_{m+1}}{1 + m + \delta_{m+1}} \quad (4.7)$$

where

$$\delta_n \geq \begin{cases} 1, & n = 1, 2, 3, \dots, m \\ n^{\frac{\delta_{m+1}}{m+1}}, & n = m + 1, m + 2, \dots \end{cases}$$

and δ_n is defined by (4.3). The estimates in (4.6) and (4.7) are sharp with the extremal function given by (4.4).

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