

CUTWIDTH OF CERTAIN CLASSES OF CIRCULANT NETWORKS

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Abstract: The cutwidth problem of a graph G is to embed G into a path such that the maximum number of edges crossing along any cut in the path is minimized. A circulant graph $C_{mk}(1, k)$ is a graph with vertex set $V = \{1, 2, \dots, mk\}$ and edge set $E = E_1 \cup E_2$, where

$$E_1 = \{(i, i + 1)\}, \quad E_2 = \{(i, (i + k) \bmod mk)\}, \quad i \in \{1, 2, 3, \dots, mk\}.$$

Bansal et al. (IEEE World Congress on Computational Intelligence, 2012) conjectured that the cutwidth of $C_{2k}(1, k)$, $3 < k \leq 30$ is 5. In this paper we not only prove this conjecture but also prove that the result is true for any $k, k > 3$.

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1. Introduction

Given a graph with n vertices, a layout on the graph is a bijection between the vertex set and the set of natural numbers from 1 to n . The cutwidth of a layout is the maximum number of edges connecting vertices on opposite sides of any of the “gaps” between successive vertices in the linear layout. The cutwidth of an embedding is the minimum of the cutwidths of layouts taken over all the possible layouts. Computing cutwidth is an NP-complete problem [1], even when the graphs are restricted to planar graphs of maximum degree three [2], unit disk graphs [3], partial grids, and consequently bipartite graphs. The problem of computing the cutwidth of a graph is known as the Minimum

Cut Arrangement problem [4]. This important graph layout problem was first proposed as a model to minimize the number of channels in a circuit [5, 6], and more recently it has found applications in areas like protein engineering [7], network reliability [8], automatic graph drawing [9], information retrieval [10], and as a subroutine in the cutting plane algorithm for TSP [11].

Polynomial-time algorithms for the exact computation of cutwidth are known only for few graph classes. Exact results are proved for various standard graphs such as paths, cycles, stars, complete graphs, hypercubes, complete bipartite graphs and graph products, 2,3-dimensional meshes, 2, 3-dimensional toroidal meshes, cylindrical 2-dimensional meshes [12], complete k -level t -ary trees, trees with diameter at most 4 and double stars, mobius ladders, caterpillars, and (m, n) -multipath [13]. Cutwidth of proper interval graphs has a trivial solution following an interval ordering of the vertices [14].

The circulant graphs are an important class of topological structures of interconnection networks. They are symmetric with simple structures and easy extendability. Circulant graphs have been used for decades in the design of computer and telecommunication networks due to their optimal fault-tolerance and routing capabilities [15]. The circulant network is a natural generalization of a double loop network, which was first considered by Wong and Copper-smith [16]. Circulant graphs also constitute the basis for designing certain data alignment networks for complex memory systems. Most of the earlier research concentrated on using the circulant graphs to build interconnection networks for distributed and parallel systems. Circulant graphs are intensively researched in computer science, graph theory and discrete mathematics [17].

In this paper we investigate the cutwidth of circulant networks $C_{2k}(1, k)$, $k > 3$ is 5.

2. Basic concepts

In this section we give the basic definitions and preliminaries related to cutwidth problems.

Given an undirected graph $G = (V, E)$ with $|V| = n$, a layout φ on G is a one-to-one function $\varphi : V \rightarrow \{1, 2, 3, \dots, n\}$. Given a layout φ on G , we define the sets $L(i, \varphi, G) = \{u \in V(G) : \varphi(u) \leq i\}$, $R(i, \varphi, G) = \{u \in V(G) : \varphi(u) > i\}$ and cut $\theta(i, \varphi, G) = |\{uv \in E(G) : u \in L(i, \varphi, G) \wedge v \in R(i, \varphi, G)\}|$.

Definition 1. [18] Given a graph G and a layout φ on G , let $cw(\varphi, G) = \max_{i=1}^n \theta(i, \varphi, G)$. Define $cw(G) = \min_{\varphi} cw(\varphi, G)$ where the minimum is taken over all linear layouts φ on G . The cutwidth problem of G is to find a layout

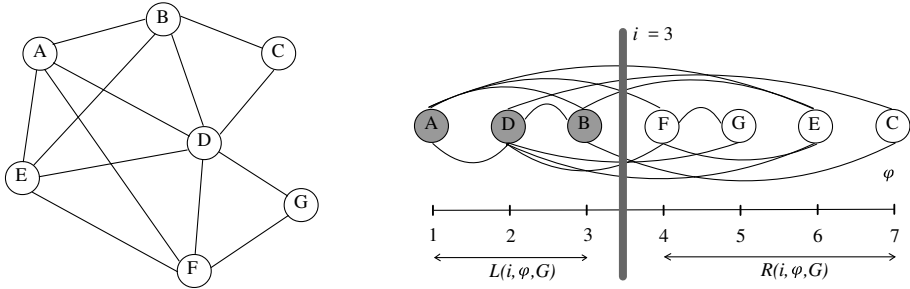


Figure 1: Graphical representation of the layout $\varphi = \{(A, 1), (B, 3), (C, 7), (D, 2), (E, 6), (F, 4), (G, 5)\}$.

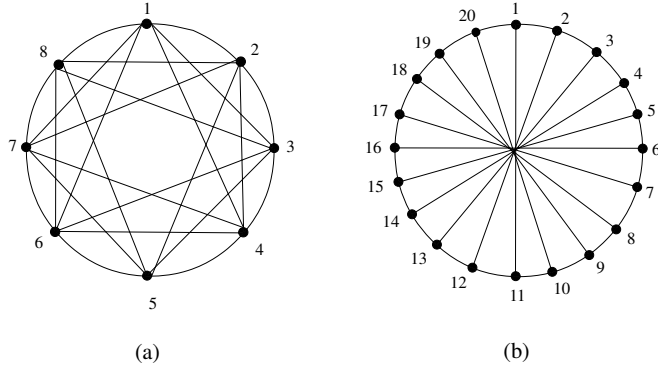


Figure 2: (a) $G(8; \pm\{1, 2, 3\})$ and (b) $C_{20}(1, 10)$

φ on G such that $cw(\varphi, G) = cw(G)$.

In Fig. 1, by drawing a vertical line between position 3 and position 4, the vertices to the left of the line belong to $L(3, \varphi, G)$ and the vertices to the right of the line belong to $R(3, \varphi, G)$. It is easy to compute the cut $\theta(3, \varphi, G)$ by counting the number of edges that cross the vertical line, in other words the number of edges that cross from $L(3, \varphi, G)$ to $R(3, \varphi, G)$.

Definition 2. [19] A circulant undirected graph, denoted by $G(n; \pm S)$ where $S \subseteq \{1, 2, \dots, \lfloor n/2 \rfloor\}$, $n \geq 3$ is defined as a graph consisting of the vertex set $V = \{1, 2, \dots, n-1\}$ and the edge set $E = \{(i, j) : |j-i| \equiv s \pmod{n}, s \in S\}$.

For convenience we label $V(G)$ as $1, 2, 3, \dots, n$. The circulant graph shown in Fig. 2(a) is $G(8; \pm\{1, 2, 3\})$. It is clear that $G(n; \pm 1)$ is the undirected cycle C_n and $G(n; \pm\{1, 2, 3, \dots, \lfloor n/2 \rfloor\})$ is the complete graph K_n . Further $G(n; \pm\{1, 2, 3, \dots, j\})$, $1 \leq j \leq \lfloor n/2 \rfloor$, $n \geq 3$ is a $2j$ -regular graph [20].

Definition 3. [21] $C_{mk}(1, k)$ is a special class of circulant networks with vertices $|V| = mk$ and edges $E = E_1 \cup E_2$ where $E_1 = \{(i, i + 1)\}$, $E_2 = \{(i, (i + k) \bmod mk)\}$, $i \in \{1, 2, 3, \dots, mk\}$.

The circulant graph shown in Fig. 2(b) is $C_{20}(1, 10)$. Clearly $C_{2k}(1)$ is a cycle on $2k$ vertices. Let the vertices be labelled $1, 2, 3, \dots, 2k$ in the clockwise sense.

3. Exact Cutwidth Of Circulant Graph $C_{2k}(1, k)$

In this section we prove that the cutwidth of $C_{2k}(1, k)$ is 5 for $k > 3$. Incidentally, this proves the conjecture proposed by Bansal et al. [21] for the restricted class $C_{2k}(1, k)$, $3 < k \leq 30$ of circulant networks.

Maximum subgraph problem [22] A maximum subgraph problem for a graph $G = (V, E)$ is to find an induced subgraph A of G on k vertices for a given integer k , with maximum number of edges of G .

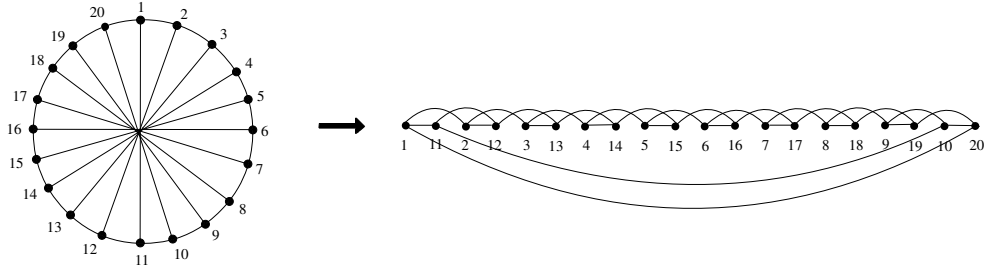
Lemma 1. (Congestion Lemma) [22] *Let G be an r -regular graph and f be an embedding of G into H . Let S be an edge cut of H such that the removal of edges of S leaves H into 2 components H_1 and H_2 and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Also S satisfies the following conditions:*

- (i) *For every edge $(a, b) \in G_i$, $i = 1, 2$, $P_f(a, b)$ has no edges in S .*
- (ii) *For every edge (a, b) in G with $a \in G_1$ and $b \in G_2$, $P_f(a, b)$ has exactly one edge in S .*
- (iii) *G_1 is an optimal set.*

Then $EC_f(S)$ is minimum and $EC_f(S) = r|V(G_1)| - 2|E(G_1)|$.

Lemma 2. $C_{2k}(1, k)$, $k > 3$ is triangle-free.

Proof. Let G be the circulant graph $C_{2k}(1, k)$, $k > 3$ with $2k$ vertices. The subgraph $C_{2k}(1)$ is a cycle on $2k$ vertices and $2k$ edges. The remaining k edges of G are chords of the cycle C . Each chord subtends two arcs of C of equal length on $k - 1$ vertices. Since $k - 1 > 2$, the chord and a vertex on the arc do not induce a triangle. Thus G is triangle-free. $\square$

Figure 3: Embedding of $C_{20}(1, 10)$ in P_{20}

Lemma 3. *The maximum subgraph of $G = C_{2k}(1, k)$, $k > 3$ on four vertices is a 4-cycle.*

Proof. Any graph on 4 vertices with more than 4 edges contains a triangle contradicting Lemma 2. $\square$

Lemma 4. *The maximum subgraph of $G = C_{2k}(1, k)$, $k > 3$ on five vertices contains five edges.*

Proof. By Lemma 3, a maximum subgraph on 4 vertices is a 4-cycle in G and is of the form $i, i+1, i+\frac{n}{2}+1, i+\frac{n}{2}$. If the fifth vertex v is of degree 2 in H , then v has label $i+2$ and a triangle is formed in H , which is not possible. Hence the degree of v should be 0 or 1. Since H is a maximum subgraph of G , degree of v must be 1. Hence H has 5 edges. $\square$

Lemma 5. *The maximum subgraph H of $G = C_{2k}(1, k)$, $k > 3$ on $2t$, $t \geq 2$ vertices has $3t-2$ edges induced by the vertices $1, 2, 3, \dots, t, k+1, k+2, \dots, k+t$ and that on $2t+1$, $t \geq 2$ vertices has $3t-1$ edges induced by the vertices $1, 2, 3, \dots, t, t+1, k+1, k+2, \dots, k+t$.*

Proof. Let H be a subgraph of G on r vertices. We prove the result by induction on r . Two cases arise:

Case 1 (r is even): When $r = 4$, the result is true by Lemma 3. Assume the result is true for $r = 2t$, $t \geq 3$. Consider two vertices u, v in G , which are at distance k apart. $H \setminus \{u, v\}$ is connected on $2t$ vertices and hence by induction hypothesis, contains $3t-2$ edges. Now include the two vertices u, v which add a maximum of three edges. Hence total number of edges is given by $3t-2+3 = 3(t+1)-2$. Thus the result is true for all $t \geq 2$.

Case 2 (r is odd): When $r = 3$, the result is true by Lemma 2. We prove the result for $t \geq 2$. Consider two vertices u, v in G , which are at the

distance k apart. $H \setminus \{u, v\}$ is connected on $2t + 1$ vertices and hence by induction hypothesis, contains $3t - 1$ edges. Now include the two vertices u, v which add a maximum of three edges. Hence total number of edges is given by $3t - 1 + 3 = 3(t + 1) - 1$. Thus the result is true for all $t \geq 2$. $\square$

Cutwidth Algorithm

Input: A circulant network $C_{2k}(1, k)$, $k > 3$ and a path P_{2k} .

Algorithm: Label the consecutive vertices of $C_{2k}(1, k)$, $k > 3$ in $C_{2k}(1, k)$, $k > 3$ as $1, 2, 3, \dots, 2k$ in the clockwise sense. Label the alternate vertices of P_{2k} as $1, 2, 3, \dots, k$ from left to right beginning from the first vertex; label the alternate vertices of P_{2k} as $k + 1, k + 2, k + 3, \dots, 2k$ from left to right beginning from the second vertex. Define embedding $\varphi : C_{2k}(1, k) \rightarrow P_{2k}$ by $\varphi(2i - 1) = i$ and $\varphi(2i) = i + k$, $1 \leq i \leq k$. See Figure 3.

Output: An embedding φ of $C_{2k}(1, k)$, $k > 3$ into P_{2k} given by $\varphi(2i - 1) = i$ and $\varphi(2i) = i + k$, $1 \leq i \leq k$ with cutwidth = 5.

Proof of correctness: Every edge $e = (i, i + 1)$, $1 \leq i \leq 2k - 1$ in P_{2k} is a cut edge. Therefore $P_{2k} \setminus e$ has two components say H_1 and H_2 such that H_1 is a path induced by $1, 2, 3, \dots, i$. By Lemma 5, $\varphi^{-1}(H_1)$ is a maximum subgraph of $C_{2k}(1, k)$. Then by Congestion Lemma 1, congestion on e is $3i - 2(\frac{3i}{2} - 2) = 4$, when i is even and congestion on e is $3i - 2(\frac{3(i-1)}{2} - 2) = 5$, when i is odd. Hence cutwidth $cw(G) = 5$.

4. Conclusion

In this paper, we compute the exact cutwidth of certain circulant graphs and prove a conjecture given by Bansal et al. The Congestion Lemma is an effective tool to compute the cutwidth of graphs. It would be interesting to study cutwidth of interconnection networks such as torus, cube connected cycle and so on.

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