International Journal of Pure and Applied Mathematics
Volume 109 No. 5 2016, 67-74
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v109i5.8



A FULLY QUADRATURE METHOD FOR THE NONLINEAR FOUR-WAVE INTERACTIONS

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Abstract: A new attempt is made to compute the nonlinear four wave interactions in the discrete spectral wave model represented by the 6-D Boltzmann integral using fully Gauss-Legendre quadrature method. An advantage of this method is that the singularities of the integral are naturally avoided. The method can be used to obtain accurate results for the nonlinear energy transfer rate. In this paper, the evaluation of the transfer integral based on the locus parameters arc length (s) and direction (θ) are considered separately using Gauss-Legendre quadrature method. Results obtained using this method for the deep water nonlinear energy transfer rate corresponding to the input standard JONSWAP wave spectra with angular distributions in the cases 2-6 of Hasselmann and Hasselmann [2] [hereafter referred as HH] are presented. A comparison study of the present 1-D nonlinear results indicate good agreement with the corresponding results of the exact WRT method [9].

AMS Subject Classification: 65D32, 76B15, 76B60, 86A05
Key Words: deep waters, Gauss Legendre quadrature method, JONSWAP spectrum, nonlinear four-wave interactions, quadruplets, wave model

Received: October 1, 2016
Published: October 23, 2016

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1. Introduction

Nonlinear four-wave interactions play a key role in the evolution of wave spectrum. They represent the exchange of wave energy between the spectral components within the energy spectrum. The wave spectrum can be predicted using wave models. Current research in the area of wave modelling focusses on numerical methods for the nonlinear interactions which can provide accurate and computationally efficient results. Although different numerical methods are available, either accuracy can be achieved with more computation time or computational efficiency with less accuracy. Some of the contributions to the computational methods available in the literature include (i) Approximate method such as Discrete Interaction Approximation (DIA) by Hasselmann and Hasselmann [3] and (ii) exact methods such as EXACT NL developed by Hasselmann and Hasselmann [2], Webb-Resio-Tracy (WRT) method by Webb [10], Tracy and Resio [7], Reduced Interaction Approximation Method (RIAM) by Masuda [5] and Gaussian Quadrature method (GQM) by Lavrenov [4]. An independent formulation for nonlinear interactions using Gauss-Legendre quadrature method can also be found in Prabhakar and Pandurangan [6]. In spite of the availability of several methods, current third generation operational wave models still use only DIA method.

2. Computation for nonlinear energy transfer term

Hasselmann [1] developed the third order nonlinear wave-wave interactions using perturbation method. This expression known as Boltzmann integral is

$$\begin{aligned}\frac{\partial N_1}{\partial t} = \iiint G(\overrightarrow{k_1},\overrightarrow{k_2},\overrightarrow{k_3},\overrightarrow{k_4})[N_1N_4(N_3-N_2)+N_2N_3(N_4-N_1)] \\ \times \delta(\overrightarrow{k_1}+\overrightarrow{k_2}-\overrightarrow{k_3}-\overrightarrow{k_4})\delta(\omega_1+\omega_2-\omega_3-\omega_4)d\overrightarrow{k_2}d\overrightarrow{k_3}d\overrightarrow{k_4}.\end{aligned} \tag{1}$$

Here, $\overrightarrow{k_i} = (k_i, \theta_i)$ and $N_i = N(\overrightarrow{k_i}) = \frac{F(\vec{k_i})}{\omega_i}$ represent the ith interacting wave number vector and action density respectively, with $F(\vec{k_i}) = F(k_i, \theta_i)$ and ω_i denoting the energy density spectrum and the angular frequency. The dispersion relation in deep water is $\omega^2 = gk$ and the product terms involving action densities denoted by $N_{1,2,3,4}$. The term $G(\overrightarrow{k_1},\overrightarrow{k_2},\overrightarrow{k_3},\overrightarrow{k_4})$ represent the coupling coefficient and is a complicated expression. The term $\delta(...)$ denotes the delta function. The contribution to the integral comes from the set of wave quadruplets satisfying the following resonance conditions

$$\begin{aligned} \overrightarrow{k_1} + \overrightarrow{k_2} &= \overrightarrow{k_3} + \overrightarrow{k_4} \\ \omega_1 + \omega_2 &= \omega_3 + \omega_4 \end{aligned} \tag{2}$$

By integrating over $\overrightarrow{k_4}$ in Eq. (1) and using the δ-function property, the Boltzmann integral can be written as

$$\frac{\partial N_1}{\partial t} = \int T(\overrightarrow{k_1}, \overrightarrow{k_3}) d\overrightarrow{k_3}, \tag{3}$$

where the transfer integral is represented by

$$T(\overrightarrow{k_1}, \overrightarrow{k_3}) = \int G \times N \times \delta(\omega_1 + \omega_2 - \omega_3 - \omega_{1+2-3}) k_2 dk_2 d\theta_2. \tag{4}$$

Now, integrating over the wave number k_2 and using the δ-function property, we have

$$T(\overrightarrow{k_1}, \overrightarrow{k_3}) = \frac{1}{\pi} \int G \times N \times \left| \frac{\partial(\omega_1 + \omega_2 - \omega_3 - \omega_{1+2-3})}{\partial k_2} \right|^{-1} d\theta_2. \tag{5}$$

The transfer integral can be expressed in a simplified form as

$$T(\overrightarrow{k_1}, \overrightarrow{k_3}) = \frac{1}{\pi} \oint_{\theta_2} G \times J \times N d\theta_2, \tag{6}$$

where J is Jacobian transformation and is defined as

$$J = \left| \frac{\partial(\omega_1 + \omega_2 - \omega_3 - \omega_{1+2-3})}{\partial k_2} \right|^{-1}. \tag{7}$$

The following expression can be found in Van Vledder [8] for the Jacobian term in terms of group velocity $\overrightarrow{c}_{g,i}$ at the wave number vector $\overrightarrow{k_i}$.

$$J = |\overrightarrow{c}_{g,2} - \overrightarrow{c}_{g,1+2-3}|^{-1}. \tag{8}$$

In terms of the magnitude group velocity $c_{g,i} = \frac{g}{2\omega_i}$,

$$J = (c_{g,2}^2 + c_{g,1+2-3}^2 - 2c_{g,2}c_{g,1+2-3}cos(\theta_2 - \theta_{1+2-3}))^{\frac{-1}{2}}. \tag{9}$$

The polar method for computing the wave resonating quadruplets and the evaluation of the line integral using quadrature method considered in Prabhakar and Pandurangan [6] is retained in this work. In addition, the present work also considers the evaluation of the transfer integral choosing the locus parameter to be the arc length. Thus, the transfer integral is evaluated using quadrature

method by considering locus parameters s and θ_2 separately. The relation between the locus parameters θ_2 and arc length s is $\theta_2 = \frac{2\pi s}{L}$. In terms of the parameter s, Eq. 6 can be written as

$$T(\overrightarrow{k_1}, \overrightarrow{k_3}) = \frac{2}{L} \oint_s G \times J \times N ds, \tag{10}$$

where L denotes the total length of the locus for an input pair $(\overrightarrow{k_1}, \overrightarrow{k_3})$. Applying Gauss-Legendre quadrature method to the above integral over the arc length $s \in [0, L]$, we have

$$T(\overrightarrow{k_1}, \overrightarrow{k_3}) \approx \sum_{i=1}^{m} \omega_i G(s_i) \times J(s_i) \times N(s_i), \tag{11}$$

where s_i denote the abscissas of the Legendre polynomial of order m and ω_i denote the associated weights.
The polar form of Eq. (3) is

$$\frac{\partial N_1}{\partial t} = \iint T(\overrightarrow{k_1}, \overrightarrow{k_3}) k_3 dk_3 d\theta_3. \tag{12}$$

The above double integral can be evaluated using Gauss-Legendre quadrature method as follows. (i) Apply quadrature method for the outer variable θ_3 between the limits $[0, 2\pi]$; (ii) Apply composite quadrature method for the radial wave number k_3 between the limits $[k_0, k_u]$, where k_0 and k_u are the minimum and maximum wave numbers, respectively. The interval $[k_0, k_u]$ is partitioned into q number of sub-intervals with logarithmic space λ.

From Eq. (12), the discretized form of the rate of change of action density at wave number vector $(k_{1,p}, \theta_{1,t})$ can be expressed as

$$\Delta N(k_{1,p}, \theta_{1,t}) \approx C \sum_{i=1}^{q} \lambda^{i-1} \left[\sum_{j=1}^{r} \left(\omega_j \sum_{n=1}^{s} \omega_n T(k_{1,p}, \theta_{1,t}, k_{3,i,j}, \theta_{3,n}) k_{3,i,j} \right) \right], \tag{13}$$

where p and t indicates the grid points of wave number and radial direction, respectively. $k_1 = k_{1,p} = \lambda^{p-1} k_0$ with $p = 1, 2, ..., q$ and $C = \frac{\pi(\lambda-1)k_0}{2}$; $k_{3,i,j}$ are the abscissas of the Gauss-Legendre polynomial of order r, in the sub-interval i, w_j are the weights associated with the abscissas, $\theta_{3,n}$ are the abscissas of the

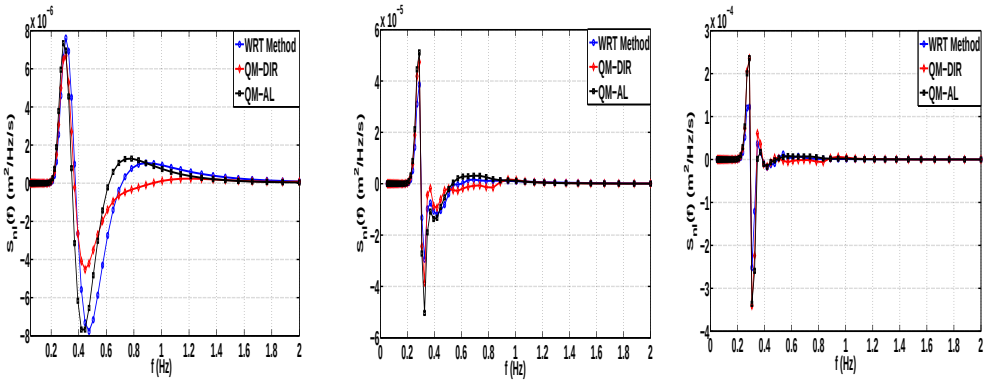
Gauss-Legendre polynomial of order s with corresponding weights ω_n. Finally the 2-D nonlinear energy transfer rate can be written as,

$$S_{nl}(f_{1,p}, \theta_{1,t}) = 4\pi k_{1,p}^2 \Delta N(k_{1,p}, \theta_{1,t}), \tag{14}$$

where $f_{1,p}$ indicate the radial frequency of the corresponding wave number $k_{1,p}$ and $\theta_{1,t} \in [0, 2\pi]$ represent the angular direction. The 1-D nonlinear energy transfer rate $S_{nl}(f)$ is determined by integrating $S_{nl}(f, \theta)$ with respect to θ.

3. Results

The computation results are illustrated for the nonlinear energy transfer term by considering an input spectrum to be the standard JONSWAP spectrum with f^{-5} tail and the shape parameters as Phillips constant $\alpha = 0.01$, peak frequency $f_m = 0.3$Hz and $\sigma = \begin{cases} 0.07, & f < f_m \\ 0.09, & f > f_m \end{cases}$. Figs. 1-3 consider the angular distributions corresponding to the cases 2-6 of HH. Here, we consider 30 quadrature points on the locus and the number of quadrature points used in the composite quadrature method are $r = 1$ point for radial wave number k_3, $s = 33$ points for angular parameter θ_3. An input polar grid consisting of 60 frequencies distributed from 0.05Hz to 2Hz, with a logarithmic spacing ($\lambda = 1.0645$); 36 directions with angular spacing ($\Delta\theta = 10^o$)is considered.



(a) (b) (c)

Figure 1: Comparison of $S_{nl}(f)$ versus frequency f, using the present quadrature method with WRT for case 2 of HH. (a) $\gamma = 1$, (b) $\gamma = 3.3$ and (c) $\gamma = 7$.

Fig. 1(a-c) shows comparison of the nonlinear energy transfer rate obtained from the fully Quadrature Method for the locus based on Arc Length (QM-AL) with exact WRT method and the fully Quadrature Method for the locus based on direction (QM-DIR) for $\gamma = 1, 3.3, 7$ respectively.

An advantage of the locus based on arc length parameter is the accuracy of the results. In the present work, the same number of quadrature points on the locus have been considered for both parameters θ_2 and s. In the 1-D nonlinear result shown in Fig. 1a for the input PM spectrum, the higher amplitude falls before the peak frequency whereas it falls after the peak in the case of exact WRT method. The results for the remaining cases in Fig. 1 agree with that of the exact WRT method. Nonetheless, the results are found to be comparable. It is also clear that the energy distribution around the peak frequency becomes narrower as γ increases.

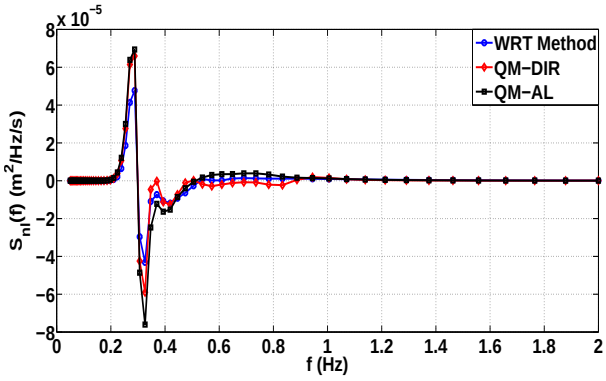


Figure 2: Comparison of the 1-D transfer rates for the case 3 of HH.

Fig. 2 shows the results of the 1-D nonlinear energy transfer rate obtained using QM-AL for the input spectrum with the angular distribution of case 3 of HH. The results are found to be comparable with the corresponding results of the WRT method for circular grid due to Vledder and Hashimoto [9] and QM-DIR.

Fig. 3 shows the the computed 1-D nonlinear energy transfer results for the cases 4, 5 and 6 of HH using the quadrature methods QM-DIR and QM-AL. Although the results are comparable, results obtained using QM-AL appears smoother than QM-DIR indicating that the parameter arc length needs to be considered for evaluating the transfer integral.

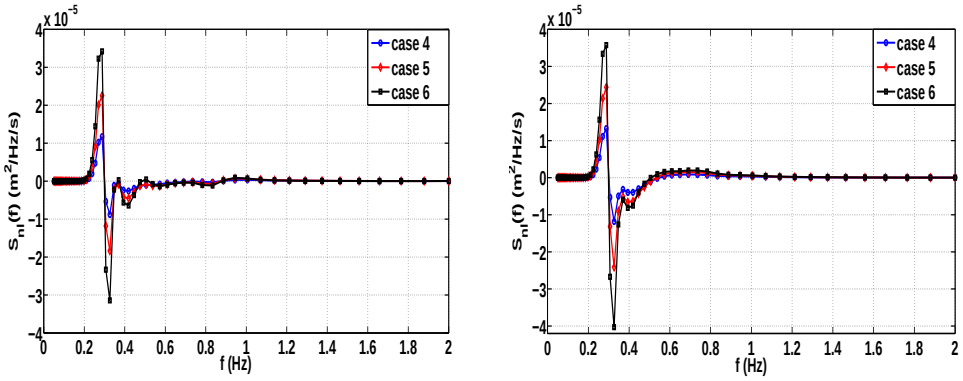


Figure 3: Results for the 1-D nonlinear energy transfer rate for different cases of HH. (a) QM-DIR and (b) QM-AL.

It is seen that the nonlinear energy transfer results have the three lobe structure (positivenegativepositive) which is explicitly shown in the Figs. 13 for both quadrature methods QM-DIR and QM-AL.

4. Conclusion

A solution procedure using Gauss-Legendre quadrature method for all the integration parameters of the Boltzmann integral is described for the computation of the nonlinear four-wave interactions in a discrete wave spectrum. Based on the choice of the parameter used for the computation of the transfer integral, two quadrature methods QM-DIR and QM-AL were considered. Results of the 1-D nonlinear energy transfer rate obtained using the quadrature methods indicate that the arc length parameter will be an appropriate choice over the direction parameter. A comparison study indicates good agreement of the results obtained using the present quadrature approach with that of the existing WRT method.

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