TOTAL EDGE IRREGULARITY STRENGTH
OF DIAMOND SNAKE AND DOVE

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Abstract: Given a graph $G(V, E)$ a labeling $\partial : V \cup E \rightarrow \{1, 2, ..., k\}$ is called an edge irregular total $k$-labeling if for every pair of distinct edges $uv$ and $xy$, $\partial(u) + \partial(uv) + \partial(v) \neq \partial(x) + \partial(xy) + \partial(y)$. The minimum $k$ for which $G$ has an edge irregular total $k$-labeling is called the total edge irregularity strength. The total edge irregular strength of $G$ is denoted by $\text{tes}(G)$. In this paper we examine certain graphs like diamond snake, dove tail graph, subdivision dove tail graphs and prove that they are total edge irregular. As our main result we prove that the bound on $\text{tes}$ is sharp for diamond snake and subdivision dove tail graph but, for dove tail graph the bound differs from the minimum $\text{tes}$ value by 1.

Key Words: Irregular total labeling, Graph labeling, irregularity strength, total edge irregularity strength, snake graphs.

1. Introduction

The area of graph theory has experienced fast developments during the last 60 years. Among the huge diversity of concepts that appear while studying this subject, one that has gained a lot of popularity is the concept of labelings of graphs. Over the past four decades or so, more than 1200 papers have appeared a bewildering array of graph labeling. This new branch of mathematics has caught the attention of many authors and many new labeling results appear every year. This popularity is not only due to the mathematical challenges of graph labeling, but also for the wide range of its applications.

Most graph labelings trace their origins to labelings presented by Alex Rosa in his 1967 paper [8] and Golomb’s research in 1972 [5]. Rosa identified three
types of labelings, which he called $\alpha$-, $\beta$-, and $\rho$-labelings. $\beta$-labelings were later renamed graceful by S.W. Golomb [5] and the name has been popular since.

Graph labelings, have often been motivated by practical considerations such as coding, X-ray crystallography, radar tracking, remote control, radio astronomy, communication networks, network flows etc.. Their theoretical applications too are numerous, not only within the theory of graphs but also in other areas of mathematics such as combinatorial number theory, linear algebra and group theory admitting a given type of labeling [4]. They are also of interest on their own right due to their abstract mathematical properties arising from various structural considerations of the underlying graphs. An enormous body of literature has grown around the theme. For a dynamic survey of various graph labelings along with an extensive bibliography, one may refer to Gallian [4].

Motivated by the notion of the irregularity strength and irregular assignments of a graph introduced by Chartrand et al. (refer [3]) in 1988 and various kinds of other total labelings, the total edge irregularity strength of a graph was introduced by Bača, Jendrol, Miller and Ryan [1] as follows: For a graph $G(V, E)$ a labeling $\partial : V \cup E \rightarrow \{1, 2, ..., k\}$ is called an edge irregular total $k$-labeling if for every pair of distinct edges $uv$ and $xy$, $\partial(u) + \partial(uv) + \partial(v) \neq \partial(x) + \partial(xy) + \partial(y)$.

The minimum $k$ for which $G$ has an edge irregular total $k$-labeling is called the total edge irregularity strength of $G$. The total edge irregular strength of $G$ is denoted by $tes(G)$.

![Diagram](image)

Figure 1: (a) Edges of $DS(3)$ (b) $tes(DS(3)) = 5$

We begin with few known results on $tes(G)$. 
Theorem 1. [1] Let $G$ be a graph with $m$ edges. Then $\text{tes}(G) \geq \lceil \frac{m+2}{3} \rceil$.

Theorem 2. [1] Let $G$ be a graph with maximum degree $\Delta$. Then $\text{tes}(G) \geq \lceil \frac{\Delta+1}{2} \rceil$.

Theorem 3. [2] A graph $G(V, E)$ of order $n$, size $m$, and maximum degree $0 < \Delta < \frac{m^{10} - 3}{\sqrt{8n}}$ satisfies $\text{tes}(G) = \lceil \frac{m+2}{3} \rceil$.

Conjecture 4. [6] For every graph $G$ with size $m$ and maximum degree $\Delta$ that is different from $K_5$, the total edge irregularity strength equals $\max\{\lceil \frac{m+2}{3} \rceil, \lceil \frac{\Delta+1}{2} \rceil\}$.

2. Main Results

2.1. Diamond snake

Definition 5. An $r$-dimensional diamond snake is a connected graph obtained from a path $P$ of length $r$ with each edge $e = (u, v)$ in $P$ replaced by a cycle of length 4 with $u$ and $v$ as nonadjacent vertices of the cycle. It is denoted by $DS(r)$.

Thus $DS(r)$ contains $r$ blocks $B_1, B_2, \ldots, B_r$, each isomorphic to a 4-cycle. The $r$-dimensional diamond snake has $(3r + 1)$ vertices and $4r$ edges.

![Figure 2: (a) $\text{tes}(DS(4)) = 6$ (b) $\text{tes}(DS(5)) = 8$](image)

Notation 6. Denote the edges of the $i^{th}$ block of $DS(r)$ as $T_L(i), B_L(i), T_R(i)$ and $B_R(i)$. See Figure 1 (a).
Lemma 7. \( \text{tes}(DS(3)) = 5. \)

Proof. Let \( DS(3) \) be labeled as in Figure 1 (b). It is easy to check that \( \text{tes}(DS(3)) = 5. \)

We now consider \( DS(r), r \geq 4. \)

**Procedure** \( \text{tes}(DS(r)) \)

**Input:** \( r \)-dimensional diamond snake, \( DS(r), r \geq 4. \)

**Algorithm:** Let \( s(i) = \left\lceil \frac{4i+2}{3} \right\rceil, \forall i \geq 4. \)

(1) Label the vertices and edges of \( DS(3) \) as in Lemma 9.

(2) Having labeled \( DS(i-1) \), label \( DS(i), i \geq 4 \) as follows.

(i) The first \( i-1 \) blocks of \( DS(i) \) induce \( DS(i-1) \) and is labeled as in \( DS(i-1) \).

(ii) To label the \( i^{th} \) block of \( DS(i) \) we proceed by labeling the unlabeled vertices of \( D(i) \) as \( s(i) \).

(iii) To label the edges we proceed as follows:

**Case 1** \( i \equiv 1 \) mod 3

\[
\begin{align*}
l(T_L(i)) &= l(T_L(i-1)) + 2 \\
l(T_R(i)) &= l(T_R(i-1)) + 2 \\
l(B_L(i)) &= l(B_L(i-1)) + 2 \\
l(B_R(i)) &= l(B_R(i-1)) + 2.
\end{align*}
\]

**Case 2** \( i \equiv 2 \) mod 3

\[
\begin{align*}
l(T_L(i)) &= l(T_L(i-1)) + 1 \\
l(T_R(i)) &= l(T_R(i-1)) \\
l(B_L(i)) &= l(B_L(i-1)) + 1 \\
l(B_R(i)) &= l(B_R(i-1)).
\end{align*}
\]

**Case 3** \( i \equiv 0 \) mod 3

\[
\begin{align*}
l(T_L(i)) &= l(T_L(i-1)) + 1 \\
l(T_R(i)) &= l(T_R(i-1)) + 2 \\
l(B_L(i)) &= l(B_L(i-1)) + 1
\end{align*}
\]

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than 1200 papers have appeared a bewildering array of graph labeling. This new branch of mathematics has caught the attention of many authors and many new labeling results appear every year. This popularity is not only due to

\[ l(B_R(i)) = l(B_R(i - 1)) + 2. \]

End Procedure \( tes(DS(r)) \).

Output: \( tes(DS(r)) = \left\lceil \frac{4r+2}{3} \right\rceil \).

Proof of Correctness:

We prove the result by induction on \( r \). By lemma 9, \( tes(DS(3)) = 5 \). This proves the result when \( r = 3 \). Assume the result for \( DS(r) \). Consider \( DS(r+1) \). Since the labeling of \( DS(r) \) is an edge irregular \( k \)-labeling, it is clear that the labeling of \( DS(r+1) \) obtained by adding a constant using step 2 (iii) is also an edge irregular \( k \)-labeling. We know by actual verification that the edge sum labels obtained in Lemma 9 are distinct. Hence the edge sum labels of the edges of \( DS(r+1) \) are also distinct.

Labeling of \( DS(4) \) and \( DS(5) \) are shown in Figure 2. For \( r < 3 \), \( DS(r) \) is total edge irregular but to give an algorithm we have considered for \( r > 3 \). Thus we have the following theorem.

**Theorem 8.** Let \( DS(r) \) be an \( r \)-dimensional diamond snake. Then \( tes(DS(r)) = \left\lceil \frac{4r+2}{3} \right\rceil \), \( r \geq 3 \).

### 2.2. Dove Tail Graph

**Definition 9.** The dove tail graph is the graph \( P_n + K_1 \), \( n \geq 2 \). The dove tail graph has \((n + 1)\) vertices and \(2n - 1\) edges. It is denoted by \( D_n \).

For \( n < 4 \), the bound for dove tail graph is sharp. But for \( n \geq 4 \) the bound is not sharp, hence we have increased the bound by 1 and given an algorithm to prove our result.

**Theorem 10.** The dove tail graph \( D_n \), \( n \geq 4 \) is total edge irregular and \( tes(D_n) = \left\lceil \frac{2n+1}{3} \right\rceil + 1 \), \( n \geq 4 \).

**Proof.**

Let \( P_n : v_1, v_2, \ldots, v_n \) be the path and let \( u_i = v_i v_{i+1} \) \((1 \leq i \leq n - 1)\) be the edges. Let \( v_{n+1} \) be the vertex of \( K_1 \) and \( t_i = v_i v_{n+1} \) \((1 \leq i \leq n)\) be the edges of \( K_1 \) joined to path \( P_n \).

For \( 1 \leq i \leq n \) define,

\[
 f(v_i) = \begin{cases} 
 i, i - 1 \text{ and so on if } i \text{ is odd} \\
 i - 1, i - 2 \text{ and so on if } i \text{ is even} 
\end{cases}
\]

and
\[
f(v_{n+1}) = \begin{cases} 
  n - 1 & \text{if } n \equiv 1 \mod 3 \\
  n & \text{otherwise}
\end{cases}
\]

For \( 1 \leq i \leq n - 1 \) define, \( f(u_i) = 1 \).

For \( 1 \leq i \leq n \) define
\[
f(t_i) = [f(v_{n-1}) + f(v_n) + f(u_{n-1})] + i - [f(v_{n+1}) + f(v_i)].
\]

From the above definition, it is clear that the edge sum labels obtained are consecutive and distinct.

Labeling of \( D_4, D_5 \) are shown in Figure 3 and labeling of \( D_6, D_7 \) are shown in Figure 4.
2.3. Subdivision Dove Tail Graph

**Definition 11.** Subdivide $D_n$ by introducing a new vertex on each edge of $D_n$. The graph so obtained is denoted by $SD_n$.

$SD_n$ has $3n$ vertices and $2(2n-1)$ edges. For $n \leq 3$, the graph $SD_n$ is total edge irregular and the bound is sharp. Here we have given the proof for $n \geq 4$.

**Theorem 12.** The subdivision dove tail graph $SD_n$, is total edge irregular and $tes(SD_n) = \lceil \frac{4n}{3} \rceil$, $n \geq 5$.

**Proof.** Let $v_1, v_2, \ldots, v_{2n-1}$ denote the vertices of the path $SD_n$ and let $u_i = v_i v_{i+1}$ ($1 \leq i \leq 2n-1$) be the edges. Let $v_{2n+1}, v_{2n+2}, \ldots, v_{3n}$ denote the vertices of the edges joining $v_1, v_3, \ldots, v_{2n-1}$ to $v_{3n}$ and $t_j = v_j v_{2n+j}$ ($i = 1, 3, 5, \ldots, 2n-1$ and $j = 1, 2, \ldots, 3n$) be the edges joining $v_{2n+1}, v_{2n+2}, \ldots, v_{3n}$ and $v_1, v_3, \ldots, v_{2n-1}$. Let $v_{2n}$ be the vertex of $K_1$ and $s_k = v_{2n+k}$ ($k = 1, 2, \ldots, n$) denote the edges joining $v_{2n+1}, v_{2n+2}, \ldots, v_{3n}$ and $v_2n$.

Define $f(v_{2n}) = tes(SD_n)$ and for $i = 1, 2, \ldots, n$ $f(v_{2n+i}) = tes(SD_n)$.

Label the vertices and edges of the path in $SD_5$ as in $SD_4$ (See Figure 5(a)). The unlabeled vertices and edges of $P_n$ are labeled following the sequence as in $SD_4$. The edges $t_j$ and $s_k$ are labeled as follows.

For $i = 1, 3, 5, \ldots, 2n-1$ and $j = 1, 2, \ldots, 3n$, define

$$f(t_j) = [f(v_{2n+j}) + f(v_j)] + j - [f(v_{2n-2}) + f(v_{2n-1}) + f(u_{2n-2})].$$

For $j = 1, 2, \ldots, 3n$ and for $k = 1, 2, \ldots, n$, define

$$f(s_k) = [f(v_{2n}) + f(v_{2n+j})] + k - [f(t_{3n}) + f(v_{3n}) + f(v_{2n-1})].$$
From the above definition, it is clear that the edge sum labels obtained are consecutive and distinct.

Labeling of $SD_5$ is shown in Figure 5(b).

3. Conclusion

In this paper, we considered certain graphs like diamond snake, dove tail graph and subdivision dove tail graph. We have proved that they are total edge irregular. This problem is under investigation for Mongolian tent graphs and Achnia graphs.

References


