m-Eternal Total Bondage Number of a Graph

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August 16, 2017

Abstract

The Eternal dominating set of a graph is defined as a set of guards located at vertices, required to protect the vertices of the graph against infinitely long sequences of attacks, such that the configuration of guards induces a dominating set at all times. The eternal m-security number is defined as the minimum number of guards to handle an arbitrary sequence of single attacks using multiple-guard shifts. Klostermeyer and Mynhardt introduced the concept of an m-eternal total dominating set, which further requires that each vertex with a guard to be adjacent to a vertex with a guard. They defined the m-eternal total domination number of a graph $G$ denoted by $\gamma_{mt}(G)$ as the minimum number of guards to handle an arbitrary sequence of single attacks using multiple guard shifts and the configuration of guards always induces a total dominating set. We define the m-Eternal Total bondage number of a graph $G$ denoted by $b_{mt}(G)$ as the minimum cardinality of set of edges $E' \subseteq E(G)$ for which $\gamma_{mt}(G - E') > \gamma_{mt}(G)$ and $G - E'$ does not contain isolated vertices. In this paper we give
the sharp bounds for $b_{mt}(G)$ and exact values of $b_{mt}(G)$ for certain classes of graphs.

AMS Subject Classification: Subject Classification

Key Words and Phrases: Eternal total domination, total domination.

1 Introduction

Let $G = (V, E)$ be a simple and connected graph of order $|V| = n$. For graph theoretic terminology we refer to Harary [5]. For any vertex $v \in V$, the open neighbourhood of $v$ is the set $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighbourhood is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighbourhood is $N[S] = N(S) \cup S$.

A set $S$ is a dominating set if $N[S] = V(G)$ or equivalently, every vertex in $V \setminus S$ is adjacent to at least one vertex in $S$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in $G$, and a dominating set $S$ of minimum cardinality is called a $\gamma$-set of $G$. A set $S \subseteq V$ is a total dominating set (TDS) of $G$ if for every $v \in V$, there exist a vertex $u \in S$, such that $uv \in E$. The total domination number of $G$ is the minimum cardinality of a TDS of $G$. It is denoted by $\gamma_t(G)$. A TDS of minimum cardinality is called a $\gamma_t$-set of $G$.

Burger et al. [1] introduced a dynamic form of domination which has been designated eternal security by Goddard et al. [4]. The eternal $m$-security number is defined as the minimum number of guards to handle an arbitrary sequence of single attacks using multiple-guard shifts.

Klostermeyer et al. [7], introduced the concept of an eternal total dominating set, which further requires that each vertex with a guard be adjacent to a vertex with a guard. It is studied in [6].

Klostermeyer et al. [7] defined an eternal dominating set (EDS) of $G$ to be a set $D$, such that for each sequence of attacks $R = r_1, r_2, \ldots$ with $r_i \in V$ there exists a sequence $D = D_1, D_2, \ldots$ of dominating sets and a sequence of vertices $s_1, s_2, \ldots$ where $s_i \in D_i \cap N[r_i]$, such that $D_{i+1} = (D_i - \{s_i\}) \cup \{r_i\}$. Note that $s_i = r_i$ is possible. The set $D_{i+1}$ is the set of locations of guards after
the attack at $r_i$ is defended. If $s_i \neq r_i$, we say that the guard at $s_i$ has moved to $r_i$. The minimum cardinality amongst all eternal dominating sets is the \textit{eternal domination number} $\gamma^\infty(G)$. For the \textit{m-eternal dominating set problem}, each $D_i$, $i \geq 1$, is required to be a dominating set, $r_i \in V$ (assuming without loss of generality $r_i \notin D_i$), and $D_{i+1}$ is obtained from $D_i$ by moving the guards to neighbouring vertices. That is each guard in $D_i$ may move to an adjacent vertex, as long as one guard moves to $r_i$. Thus it is required that $r_i \in D_{i+1}$. The size of a smallest \textit{m-eternal dominating set} (defined similar to an eternal dominating set) of $G$ is the \textit{m-eternal domination number} $\gamma^\infty_m(G)$. This ”multiple guards move” version of the problem was introduced by Goddard et al. [2, 4]. Klostermeyer et al. [8] defined an \textit{m-eternal total dominating set} (m-ETDS) of $G$ to be an EDS except that all the sets $D_i$ are total dominating sets. The minimum cardinality amongst all m-ETDSs is the \textit{m-eternal total domination number} $\gamma^\infty_{mt}(G)$. A m-ETDS of minimum cardinality is called a $\gamma^\infty_{mt}$-set of $G$. More work has been done related to this paper in [7, 9, 10, 11].

Fink et al. [3] initiated the study of bondage number of a graph $G$, where the \textit{bondage number} $b(G)$ was defined to be cardinality of the smallest number of edges $F \subseteq E(G)$ such that $\gamma(G - F) > \gamma(G)$ and sharp bounds were obtained for $b(G)$ and exact values were determined for several classes of graphs. We define \textit{m-Eternal total bondage number} $b^\infty_{mt}(G)$ to be the minimum cardinality of set of edges $E' \subseteq E(G)$ for which $\gamma^\infty_{mt}(G - E') > \gamma^\infty_{mt}(G)$ and $G - E'$ does not contain isolated vertices. In this paper we give the sharp bounds for $b^\infty_{mt}(G)$ and exact values of $b^\infty_{mt}(G)$ for certain classes of graphs.

## 2 m-Eternal Total Bondage number

In this section, we find sharp bounds for $b^\infty_{mt}(G)$ in terms of minimum degree and order of the graph. The value of $b^\infty_{mt}(G)$ for certain classes of graphs are also determined. The definition of a \textit{sole private neighbour} of a vertex $v \in S$ as defined in [12] is stated as follows.

**Definition 1.** Let $S$ be a $\gamma^\infty_{mt}$-set of a graph $G$ and $v \in S$. A vertex $w \in N(v) \cap (V - S)$ is said to be a \textit{sole private neighbour} of

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**International Journal of Pure and Applied Mathematics**

**Special Issue**
v, if v is the only member in S which can respond to an attack at w. We define the sole private neighbourhood spn(v, S) of a vertex v ∈ S as spn(v, S) = {w ∈ N(v) ∩ (V \ S) : w is a sole private neighbour of v}, such that (S \ {v}) ∪ {w} form an m- eternal total dominating set of G.

**Theorem 2.** [10] For Paths $P_n$, $n \geq 4$, $\gamma_m^\infty(P_n) = \lceil \frac{n}{2} \rceil + 1$, $n > 1$.

**Theorem 3.** [10] For cycles $C_n$,

$$\gamma_t(C_n) = \gamma_m^\infty(C_n) = \begin{cases} \frac{n}{2} + 1, & n \equiv 2 \pmod{4} \\ \frac{n}{2}, & \text{otherwise.} \end{cases}$$

**Theorem 4.** [10] For a complete $k$-partite graph, $G = K_{m_1,m_2,...,m_k}$, $m_1 \geq m_2 \geq \ldots \geq m_k$, $m_k \geq 1$, $k \geq 2$, $\gamma_m^\infty(G) = 2$.

**Theorem 5.** [10] For a wheel graph G, $\gamma_m^\infty(G) = 2$.

**Theorem 6.** For any graph G, $b_m(G) \leq \delta(G), \delta(G) > 1$.

Proof. Let $\deg(v) = \delta(G)$, $\delta(G) > 1$, and S be a $\gamma_m^\infty$- set of G, such that $v \in S$. Let $E_v$ be the set of all edges incident with v. If $\gamma_m^\infty(G) < \gamma_m^\infty(G - (E_v - e))$, where e is an edge incident with v, then clearly $b_m(G) \leq \delta(G) - 1$. Suppose that $\gamma_m^\infty(G) = \gamma_m^\infty(G - (E_v - e))$.

**Case (i):** If $spn(v, S) = \Phi$, there exists a $\gamma_m^\infty$- set say $S_1$, such that v, w ∈ $S_1$, where w ∈ $N(v)$ and $spn(v, S_1) = \Phi$. Hence $spn(w, S_1) \neq \Phi$. Otherwise $S_1 - \{v\}$ or $S_1 - \{w\}$ will be a $\gamma_m^\infty$-set of G, contradicting the minimality of S. Now z ∈ $spn(w, S_1)$. Then $\gamma_m^\infty(G) < \gamma_m^\infty(G - E_v + e - wz)$. Therefore $b_m(G) \leq \delta(G)$.

**Case (ii):** If $spn(v, S) \neq \Phi$, then $\gamma_m^\infty(G) < \gamma_m^\infty(G - E_v + e)$. Hence $b_m(G) \leq \delta(G) - 1$. \hfill \square

**Theorem 7.** If G is a connected graph of order n, then $b_m(G) \leq n - 1$.

Proof. If $\delta(G) = n - 1$, then G ≅ $K_n$ and $b_m(G) = n - 1$. If $\delta(G) < n - 1$, then by Theorem 6, $b_m(G) \leq n - 1$. Hence the result. \hfill \square

**Theorem 8.** If G is a connected graph, then $b_m(G) \leq \min\{\deg(u) + \deg(v) - 2\}$, where u and v are adjacent.
Proof. Let \( l = \min\{\deg(u) + \deg(v) - 2\} \), where \( u \) and \( v \) are adjacent. Assume that \( b_{mt}(G) > l \). If \( E_1 \) denote the set of all edges incident with at least one of \( u \) or \( v \) and \( E_2 = E_1 \setminus uv \), then \( |E_2| = l \). Since \( b_{mt}(G) > l \), we have \( \gamma_{mt}^\infty(G - E_2) = \gamma_{mt}^\infty(G) \). Now \( uv \) is a \( K_2 \) component in \( G - E_2 \). Hence \( \gamma_{mt}^\infty(G - u - v) = \gamma_{mt}^\infty(G) - 2 \). Now, choose a \( \gamma_{mt}^\infty \)-set \( S \) of \( G - u - v \), such that \( S \) contains at least one vertex adjacent to \( u \) or \( v \) in \( G \). Without loss of generality, let \( S \) contain a vertex adjacent to \( v \) in \( G \). Then the set \( S \cup \{v\} \) is a \( \gamma_{mt}^\infty \)-set of \( G \) with cardinality \( \gamma_{mt}^\infty(G) - 1 \), which is a contradiction. Hence \( b_{mt}(G) \leq l \).

Theorem 9. For wheels \( W_n, n \geq 4 \), \( b_{mt}(W_n) = \begin{cases} 2, & n = 4 \\ 1, & n \geq 5 \end{cases} \)

Proof. Let \( G = W_n \) and \( V(G) = \{v, v_1, v_2, \ldots, v_n\} \) where \( \deg(v) = n - 1 \) and \( \deg(v_i) = 3, 1 \leq i \leq n - 1 \). By Theorem 5, \( \gamma_{mt}^\infty(G) = 2 \).

Suppose that \( n \geq 5 \). Consider \( G - e \), where \( e = vv_1 \). Clearly \( \gamma_t(G - e) = 2 \). Hence \( \gamma_{mt}^\infty(G - e) \geq 2 \). Any \( \gamma_t \)-set of \( G - e \) will contain \( v \) and either \( v_2 \) or \( v_n \). Without loss of generality, consider the \( \gamma_t \)-set \( S = \{v, v_2\} \) of \( G - e \). Suppose that \( S \) is a \( \gamma_{mt}^\infty \)-set of \( G - e \), then to defend an attack at \( v_1 \) the guard at \( v \) moves to \( v_4 \) and the guard at \( v_2 \) moves to \( v \). We see that the resulting configuration of guards is not a dominating set, which is a contradiction. Hence \( \gamma_{mt}^\infty(G - e) \geq 3 \). Therefore \( b_{mt}(G) = 1 \).

Suppose that \( n = 4 \). Consider \( G - e \). If \( e = vv_1 \), then \( \gamma_t(G - e) = 2 \). Now, we claim that \( \gamma_{mt}^\infty(G - e) = 2 \). Any \( \gamma_t \)-set of \( G - e \) will contain \( v \) and either \( v_2 \) or \( v_4 \). Without loss of generality, consider the \( \gamma_t \)-set \( S = \{v_1, v_2\} \) of \( G - e \). We claim that \( S \) is a \( \gamma_{mt}^\infty \)-set of \( G - e \). In response to any attack the configuration of the guards will always form any one of the following total dominating sets \( \{v, v_2\}, \{v, v_4\}, \{v_2, v_3\} \) and \( \{v_4, v_3\} \). Hence \( S \) can eternally respond to any sequence of attacks in \( G \). Hence \( \gamma_{mt}^\infty(G - e) = 2 \). If \( e = vv_i, 2 \leq i \leq 4 \), a similar argument holds. Suppose that \( e \) is an edge on the rim of wheel. Without loss of generality, let \( e = v_1v_2 \). Now clearly \( S = \{v, v_2\} \) is a \( \gamma_{mt}^\infty \)-set of \( G - e \), as in response to any attack a guard will always be stationed at \( v \). Therefore \( \gamma_{mt}^\infty(G - e) = 2 \). Hence \( b_{mt}(G) \geq 2 \).

Let \( e_1 = vv_1, e_2 = vv_2 \) and \( G' = G - \{e_1, e_2\} \). The two possible \( \gamma_t \)-sets of \( G' \) are \( \{v_1, v_4\} \) and \( \{v_2, v_3\} \). Consider the \( \gamma_t \)-set \( \{v_1, v_4\} \)

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of $G'$. To defend an attack at $v$, the guard at $v_4$ moves to $v$ and the guard at $v_1$ moves to $v_4$. We see that the resulting configuration of guards is not a dominating set, which is a contradiction. Therefore $\gamma_{\text{mt}}^\infty(G') > \gamma_{\text{mt}}^\infty(G)$. Hence $b_{\text{mt}}(G) = 2$. 

**Theorem 10.** For paths $P_n$, $b_{\text{mt}}(P_n) = 1$.

**Proof.** By Theorem 2, $\gamma_{\text{mt}}^\infty(P_n) = \lceil \frac{n}{2} \rceil + 1$. For any edge $e$ of $P_n$, $P_n - e \cong P_k \cup P_{n-k}$, $k \geq 2$.

If $k$ is even, then $n-k$ is odd.

$$\gamma_{\text{mt}}^\infty(P_k) + \gamma_{\text{mt}}^\infty(P_{n-k}) = \frac{k}{2} + 1 + \frac{n-k+1}{2} + 1$$

$$= \frac{n}{2} + \frac{5}{2}$$

$$> \gamma_{\text{mt}}^\infty(P_n).$$

Hence $\gamma_{\text{mt}}^\infty(P_n - e) > \gamma_{\text{mt}}^\infty(P_n)$. Therefore $b_{\text{mt}}(P_n) = 1$.

If $k$ is odd, then $n-k$ is even.

$$\gamma_{\text{mt}}^\infty(P_k) + \gamma_{\text{mt}}^\infty(P_{n-k}) = \frac{k+1}{2} + 1 + \frac{n-k}{2} + 1$$

$$= \frac{n}{2} + \frac{5}{2}$$

$$> \gamma_{\text{mt}}^\infty(P_n).$$

Hence $\gamma_{\text{mt}}^\infty(P_n - e) > \gamma_{\text{mt}}^\infty(P_n)$. Therefore $b_{\text{mt}}(P_n) = 1$. 

**Theorem 11.** For Cycles $C_n$, $b_{\text{mt}}(C_n) = \begin{cases} 2, & n \equiv 2 \pmod{4} \\ 1, & \text{otherwise} \end{cases}$.

**Proof.** By Theorem 3, $\gamma_t(C_n) = \gamma_{\text{mt}}^\infty(C_n) = \begin{cases} \frac{n}{2} + 1, & n \equiv 2 \pmod{4} \\ \left\lceil \frac{n}{2} \right\rceil, & \text{otherwise}. \end{cases}$

For any edge $e$ of $C_n$, $C_n - e = P_n$.

**Case (i):** $n \equiv 2 \pmod{4}$

$$\gamma_{\text{mt}}^\infty(C_n - e) = \gamma_{\text{mt}}^\infty(P_n) = \frac{n}{2} + 1$$

$$= \gamma_{\text{mt}}^\infty(C_n).$$
Since $\gamma_{\infty}^\infty(C_n - e) = \gamma_{\infty}^\infty(C_n)$, $b_{mt}(C_n) \geq 2$.
Removal of two edges from $C_n$ will result into two components $P_k$ and $P_{n-k}$, $k \geq 2$.
Without loss of generality let $k$ be even, $n - k$ be odd. Then

$$\gamma_{\infty}^\infty(P_k) + \gamma_{\infty}^\infty(P_{n-k}) = \frac{k}{2} + 1 + \frac{n-k+1}{2} + 1$$
$$= \frac{n}{2} + \frac{5}{2}$$
$$> \gamma_{\infty}^\infty(C_n).$$

Hence $b_{mt}(C_n) = 2$.

**Case (ii):** $n \not\equiv 0 (\text{mod } 4)$

$$\gamma_{\infty}^\infty(C_n - e) = \gamma_{\infty}^\infty(P_n) = \left\lceil \frac{n}{2} \right\rceil + 1$$
$$= \frac{n}{2} + 1$$
$$> \gamma_{\infty}^\infty(C_n).$$

Hence $b_{mt}(C_n) = 1$.

**Theorem 12.** For a complete bipartite graph $K_{r,s}$, $r, s \geq 2$, $b_{mt}(K_{r,s}) = 1$.

**Proof.** Let $G = K_{r,s}$, $V_1 = \{v_1, v_2, \ldots, v_r\}$ and $V_2 = \{u_1, u_2, \ldots, u_s\}$ be the bipartition of $V(G)$. By Theorem 4, $\gamma_{\infty}^\infty(G) = \gamma_t(G) = 2$.
Now, let $e = v_i u_i$, then $\gamma_t(G - e) = 2$. Let $S = \{v_i, u_j\}, i \neq j$ be a $\gamma_t$-set of $G - e$. To defend an attack at $u_i$, the guard at $v_j$ move to $u_i$ and the guard at $u_j$ moves to $v_j$, leaving $v_i$ undefended. Hence $\gamma_{\infty}^\infty(G - e) > \gamma_{\infty}^\infty(G)$. Therefore $b_{mt}(G) = 1$.

**References**


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